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ALGEBRA.

CHAPTER I.

ELEMENTARY PRINCIPLES.

1. Algebra treats of numbers, the numbers being represented by letters (symbols of *quantity*), affected with certain symbols of *quality*, and connected by symbols of *operation*.

It is easy to see that these symbols of quantity may be dealt with very much as we deal with concrete quantities in arithmetic. Thus, allowing the letter a to stand for the number of units in any quantity, and allowing also $2a$, $3a$, $4a$, &c., to stand respectively for *twice*, *thrice*, *four times*, &c., as large a quantity as the letter a , it at once follows that we may perform the operations of addition, subtraction, multiplication, and division upon these symbols exactly as we do in ordinary arithmetic upon concrete quantities. For instance, $4a$ and $6a$ make $10a$, $9a$ exceeds $5a$ by $4a$, $15a$ is 5 times $3a$, and $7a$ is contained 8 times in $56a$.

Neither is it necessary in these operations to *state*, or even to *know* the exact number of units for which any symbol of quantity stands, nor indeed the nature of these units; it is simply sufficient that it is a symbol of *quantity*. Thus, in the science of chemistry, we use a weight called a *crith*; and a person unacquainted with chemistry might not know whether a *crith* were a measure of length, weight, or capacity, or indeed whether it were a measure at all, yet he would at once allow that 6 *criths* and 5 *criths* are 11 *criths*, that twice 4 *criths* are 8 *criths*, &c.

The Signs + and - as Symbols of Operation.

2. In purely arithmetical operations, the signs + and - are respectively the signs of addition and subtraction. In this sense, too, they are used in algebra.

Thus, $a + b$ means that b is to be *added* to a , and $a - b$ means that b is to be *subtracted* from a .

Hence, as long as a and b represent ordinary arithmetical numbers, $a + b$ admits of easy interpretation, as also does $a - b$, when b is not greater than a . But when b is greater than a , the expression $a - b$ has no arithmetical meaning. By an extension, however, of the use of the signs + and -, we are able to give such expressions an intelligible signification, whatever may be the quantities represented by a and b .

Positive and Negative Quantities.—The Signs + and - as Symbols of Affection or Quality.

3. DEF.—A *positive* quantity is one which is affected with a + sign, and a *negative* quantity is one which is affected with a - sign.

Let BA be a straight line, and O a point in the line; and suppose a person, starting from O, to walk a miles in the direction OA. Suppose also another person, starting from the same or any other point in BA to walk a miles in the direction OB. These persons will thus walk a miles each in *exactly opposite* directions. Now, we call one of these directions *positive* (it matters not which) and the other *negative*. Let us take the direction OA as *positive*. We then have the first person walking a miles in a *positive* direction, and the second walking a miles in a *negative* direction. We represent these distances algebraically by $+a$ and $-a$ respectively.

It will therefore be seen that the signs + and - have no effect upon the magnitudes of quantities, but that they express the *quality* or *affection* of the quantities before which they stand.

Again, suppose a person in business to get a profit of £6, while another suffers a loss of £6. We may express these facts algebraically in two ways. We may consider *gain* as positive, and *loss* as negative gain, and say that the former has gained + 6 pounds, while the latter has gained - 6 pounds. Or we may consider *loss* as positive, and *gain* as negative loss, and say that the former has lost - 6 pounds, while the latter has lost + 6 pounds. We hence see that the gain of + 6 is equal to a loss of - 6, and that a gain of - 6 is equal to a loss of + 6.

The Sum of Algebraical Quantities.

4. Let a distance AB be measured to the *right* along the line AX. And let a further distance BC be measured from B in the *same* direction. * By the *sum* of these lines we mean the resulting distance of the point C from the original point A, that is to say, the distance AC.

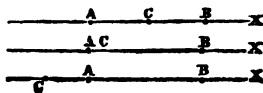
(It may be remarked that we *add* the line BC to the line AB by measuring BC in its *own proper direction* from the extremity B of AB. It is hardly necessary to remind the student that both lines are in the same straight line AX.)

Let us represent the distances AB and BC by + a and + b respectively; then the *algebraical sum* of the lines will be represented writing these quantities side by side, each with its *own proper sign* of affection.

Thus the sum of the distances AB and BC is expressed by + $a + b$, or, as it is usual to omit the + sign of a positive quantity when the quantity stands alone or at the head of an algebraical expression, the sum of AB and BC is expressed by $a + b$.

Hence, the interpretation of $a + b$ is that it represents the distance AC.

Again, taking as above + a to represent the distance AB along the straight line AX, and measured to the *right*, let a distance BC be measured from B in the same straight line AX, but this time to the *left*.



Let the latter distance be represented by $-b$.

Then, on the principle above, AC is the *sum* of these distances, and this *sum* is represented algebraically by $+a - b$ or $a - b$.

(It will be seen that the distance BC is again measured from B in its *own proper direction*, and that the resultant distance AC of the point C is again the *sum* of the line AB and BC.)

There will evidently be three cases, viz. :—

1. When the distance BC is less in magnitude than AB, in which case the point C is on the right of A, and the distance AC is *positive*.

2. When the distance BC is *equal* in magnitude to AB, in which case the point C coincides with A, and the distance AC is *zero*.

3. When the distance BC is *greater* in magnitude than AB, the point C being then on the left of A, and the distance is *negative*.

Now, $a - b$ in all these cases represents the distance AC. It therefore admits of intelligible interpretation whether b be less than, equal to, or greater than a .

And, since the distance AC is obtained in the first two cases by subtracting the distance BC from that of AB, and in the second case by subtracting as far as AB will allow of subtraction, and measuring the remainder to be subtracted in an *opposite* direction, it follows that—

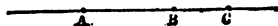
The sign $-$, which, standing before a letter, is a symbol of quality, becomes at once a symbol of subtraction in all cases when the quantity in question is placed immediately after any other given quantity with its proper sign of affection.

Hence also we may conclude that the addition of a *negative* quantity is equivalent to the subtraction of the corresponding *positive* quantity.

5. We may prove in a similar way that—

The subtraction of a *negative* quantity is equivalent to the addition of the corresponding *positive* quantity.

Let, as before, $+a$ represent the distance AB, measured from the point A to the right, and let it be required to subtract from $+a$ the distance represented by $-b$.



Now, in the last article, we *added* a distance to a given distance AB, by measuring the second distance in its *own direction* from the extremity B. We shall therefore be consistent if we *subtract* a given distance $-b$ by measuring this distance in a direction *exactly opposite* to its own direction, from the same extremity B.

Now, the direction of $-b$ is to the *left*. If, therefore, we measure a distance BC to the *right*, equal in magnitude to the distance to be subtracted, we obtain a distance AC which is correctly represented by $a - (-b)$. But AC is also correctly represented by $a + b$, and hence it follows that $a - (-b) = a + b$.

We may apply the above principle to all magnitudes which admit of continuous and indefinite extension; as, for instance, to forces which pull and push, attract and repel; to time past and time to come, to temperatures above zero and below zero, to money due and money owed, to distance up and distance down, &c., in all which cases, having represented one by a quantity affected with a $+$ sign, we may represent the other by a quantity affected with a $-$ sign.

6. *In expressing the sum of a number of quantities, the order of the terms is immaterial.*

We will take, as our illustration, a body subject to various alterations of temperature, and we will suppose the temperature of the body, before the changes in question, to be zero or 0° . Let the temperature now undergo the following changes—viz., a rise of a° , a fall of b° , a fall of c° , and, lastly, a rise of d° . Let us consider a *rise* as positive, and therefore a *fall* as negative. We may then represent these changes respectively by $+a$, $-b$, $-c$, $+d$.

And it is further evident that the resulting temperature will be represented by the *sum* of these quantities, which, as previously written, will be $a - b - c + d$.

But again, it is plain that the resulting temperature of the body will not be affected if these changes of temperature take place in the reverse order, or in any other order. Thus, suppose the temperature first falls c° , then rises a° , then rises d° , and, lastly, falls b° , it is evident that the final temperature will be the same as before. And the *sum* of the quantities $-c + a + d - b$, represents this final temperature.

Now, expressing the sum of these quantities by writing them (Art. 4) side by side with their proper signs of affection, and in the order in which they stand, we have $-c + a + d - b$ for the *sum*.

It therefore follows, since we might have chosen any other order of these terms with a similar result, that the sum of any number of quantities, $+a, -b, -c, +d$, may be expressed by writing the terms side by side with their proper signs of affection *in any order whatever*.

Nevertheless, for convenience, and for other reasons, we write the terms *generally* in *alphabetical order*, or we arrange them according to the *power* (see Art. 16) of some particular letter.

7. We may sum up the results and remarks of the last four articles as follows:—

1. *Positive* and *negative* are used in exactly opposite senses.

2. The sign $+$ before an algebraical quantity *affirms* the quality of the quantity as represented without the sign in question.

Thus, $+(+a) = +a$, and $+(-b) = -b$.

3. The sign $-$ before an algebraical quantity *reverses* the quality of the quantity as represented without the sign in question.

Thus, $-(+a) = -a$, and $-(-b) = +b$.

4. The algebraical subtraction of a quantity is the same as the addition of the quantity with the sign of affection *reversed*.

5. The algebraical addition of quantities is expressed by writing the quantities down side by side with their proper signs of affection. And they *may* be written down in any order, though we generally write them in *alphabetical order*, or arranged according to the *power* (Art. 16) of some letter.

Brackets.

8. Brackets— $()$, $\{\}$, $[\]$ —are used, for the most part, whenever we wish to consider an algebraical expression containing more than one term *as a whole*.

Thus, if we wish to express that the quantity $3a + 7b$ is to be added as a whole to $4a$, we write—

$$4a + (3a + 7b),$$

and, while inclosed within brackets, we think and speak of $3a + 7b$ as *one quantity*.

Again, if we wish to express that $b - c$ is to be subtracted from a , we write—

$$a - (b - c).$$

Let us consider what is the result of subtracting $(b - c)$ from a . We may evidently, if we please, subtract b first, then afterwards $-c$ from the quantity so obtained, without affecting this result.

Now, we know by Art. 7 (4.) that this is equivalent to adding the quantities $-b$ and $+c$ successively.

Now, the sum of a , $-b$, $+c$, is $a - b + c$.

We have therefore $a - (b - c) = a - b + c$.

We observe that the sign of b *within* the brackets is $+$, and that of c is $-$, whereas, in our final result, these signs are both reversed. And we hence arrive at the following important principle:—

When a *minus* sign stands before a bracket, its effect on removing the brackets is to *reverse* the sign of affection of every term *within*.

And it is evident that we may, by a similar course of reasoning, arrive at a principle equally important, viz:—

When a *plus* sign stands before a bracket, its effect on removing the bracket is to *affirm* the sign of affection of every term within.

We shall show, in Art. 9, the use of brackets in expressing the product or quotient of quantities.

Though, as stated above, brackets are, for the most part, used to group together as a whole a number of quantities, they are sometimes used to inclose single terms. Thus, in Art. 5, we have the expression $a - (-b)$. Now, the brackets are used here to express that the *negative quantity* is to be subtracted as a *negative quantity*. And, in the same way, the expression $a + (-b)$ indicates that the *negative quantity* $(-b)$ is to be *added* to the quantity a . When one pair only is required we generally use the brackets $()$; if,

however, a quantity already in brackets is to be inclosed in a second pair, we use $\{ \}$, as in the expression—

$$3a - \{6b + (4c - d)\}.$$

If a third pair be required we use the brackets $[]$, and finally, we sometimes find it convenient to group a number of terms by means of a *vinculum*, thus—

$$4x - [6x - \{5y + (3z - \overline{7x - y})\} + 2y].$$

It must be remembered that the *vinculum* has in an expression exactly the same force as brackets.

9. We shall, in this Article, show how to find the value of a few algebraical expressions, as illustrations of the foregoing principles:—

Ex. 1.—If $a = 1$, $b = 2$, $c = 4$, find the value of $3a + 5b + 7c$.

We have only to substitute the value of the letters in the given expression, putting a sign of multiplication to avoid ambiguity.

Thus we have—

$$\begin{aligned} 3a + 5b + 7c &= 3 \times 1 + 5 \times 2 + 7 \times 4. \\ &= 3 + 10 + 28 = 41. \end{aligned}$$

Ex. 2.—If $x = 5$, $y = 2$, $z = 6$, find the value of $4x + 3y - 9z$.

$$\begin{aligned} \text{Here, } 4x + 3y - 9z &= 4 \times 5 + 3 \times 2 - 9 \times 6 \\ &= 20 + 6 - 54 \\ &= 26 - 54. \end{aligned}$$

The negative quantity is here the larger, and exceeds the positive quantity by 28, and hence (Art. 4 (3),) the result will be *negative*.

We therefore have—

$$4x + 3y - 9z = -28.$$

Ex. 3.—If $x = 1$, $y = 3$, $z = 0$, find the value of—

$$3x - \{2y + (6z - \overline{5x - y})\}.$$

The given expression—

$$\begin{aligned} &= 3 \times 1 - \{2 \times 3 + (6 \times 0 - \overline{5 \times 1 - 3})\} \\ &= 3 - \{6 + (0 - \overline{5 - 3})\} \end{aligned}$$

$$\begin{aligned}
 &= 3 - \{ 6 + (0 - 2) \} . \\
 &= 3 - (6 - 2) = 3 - 4 = -1.
 \end{aligned}$$

It may be advisable to simplify expressions of this kind before substituting the given value of the letters. The method of doing this, however, is deferred till we come to Chapter II.

Ex. I.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, $e = 4$, find the value of the following :—

1. $4a + 2b$, $3b + 7c$, $6a + 4d$, $4c - 7e$.
2. $a + b + c$, $a - b + c$, $b + c - a$, $a + b - c$.
3. $3a + 7b - 4c$, $2a + 7d + 3c$, $7a - 10b + 2c$.
4. $6a - (2b - 3c)$, $7b + (2a - 4d)$.
5. $3c + (6a - 7b + c)$, $2b - \{ 7c + (4d - b) \}$.
6. $\{ 7b - (2d - c) \} - (4b - c + e)$.

If $x = 2$, $y = 3$, $z = 4$, find the sum and difference of the following expressions :—

7. $3x - 4y$ and $3y - 4x$.
8. $7(x - y)$ and $4(y - z)$.
9. $3x - (7y + 4z)$ and $8y + 5z - 3x$.
10. $12y - \overline{6y + 4z}$ and $12y + \overline{6y + 4z}$.
11. $x - y - z$ and $x - (y - z)$.
12. $x - (-3y)$ and $y - \{ 6x - (-3z) \}$.

Product of Two or more Quantities.

10. The product of two or more quantities may be expressed in several ways.

Thus, the product of a , b , c may be written as follows :—

1. abc , by placing the letters side by side without *any sign* between them.
2. $a \times b \times c$, by placing between them the sign $\times$.
3. $a . b . c$, by placing a dot between them.

4. $(a)(b)(c)$, by inclosing the quantities in brackets, writing them without any sign between the brackets.

When, however, the quantities are negative, or either of them consists of more than one term, it is *best* to inclose such quantities in brackets, and, in most cases, it is *necessary* to do so.

Thus, the product of $a, -b, c$ would not be correctly expressed by $a - bc$ (for this means, that the product of b and c is to be subtracted from a), but must be written $a(-b)c$. Again, the product of $2a + 3b$ and $a + 5b$ cannot be written $2a + 3ba + 5b$, as this expression means that three times the product of $b \times a$ is to be added to $2a$, and then $5b$ to the result. The product is correctly expressed thus—

$$(2a + 3b)(a + 5b).$$

(The student cannot be too strongly cautioned against leaving out brackets in cases of this kind.)

The Order of the Letters.

11. It is evident that a times $b = b$ times a ; for, if we arrange a rows of b things so as to have a horizontal rows and b vertical columns, we may either consider the number of *rows* or the number of *columns*. In the former case we have a times b things, and, in the latter, b times a things.

We may therefore write the letters whose product we wish to express in any order.

Thus, the product of a, b, c may be written in either of the following ways:—

$$abc, acb, bac, cba, bca, cab.$$

For convenience, however, and for reasons which the student will see as he proceeds with the subject, we write them in the order of the alphabet, unless there happen to be special reasons to the contrary.

Rule of Signs in Multiplication.

12. It was shown (Art. 3) that a *minus* sign does not affect the magnitude of a quantity, but simply its *affection* or

quality. It therefore follows that the product of $+a$ and $-b$ will be the same in magnitude as that of $+a$ and $+b$, or of $a \times b$ —that is, will be equal in magnitude to ab . But it is evident that the quality of the product will be the same as that of $-b$, for it is a times a *negative* quantity.

We therefore have $+a(-b) = -ab$.

So also, the product of $-a$ and $+b$, will have the same magnitude as that of $+a$ and $+b$ —that is, its magnitude will be ab . But since, to take a geometrical illustration, a line of length, b , drawn *negatively once, negatively twice, &c.*, must give a negative result, the quality of the product in question must be negative.

We therefore have $(-a)(+b) = -ab$.

Again, the product of $-a$ and $-b$ will be equal in magnitude to ab , and its quality will be evidently the *reverse* of the quality of the product of $-a$ and $+b$, and the quality of the latter product is above shown to be *negative*. The product of $-a \times -b$ will be therefore positive.

Hence we have $(-a)(-b) = +ab$.

Collecting these results, and remembering that $(+a)(+b) = +ab$, we have—

$$\begin{aligned} (+a)(+b) &= +ab. \\ (+a)(-b) &= -ab. \\ (-a)(+b) &= -ab. \\ (-a)(-b) &= +ab. \end{aligned}$$

We have then the following rule of signs in multiplication :—

RULE.—Like signs give $+$, and unlike signs give $-$.

Quotient of Two Quantities.

13. The quotient of two quantities is expressed in either of the two following ways :—

1. By placing the divisor under the dividend, separated by a line.

Thus, the quotient of a by $b = \frac{a}{b}$.

2. By placing the divisor after the dividend with a sign $\div$ between them ; thus, $a \div b$.

When either of the quantities is negative, it is better, if the second be used, to inclose the negative quantity, if the divisor, in brackets.

Thus, the quotient of a by $-b$ may be written $a \div -b$, but it is better written $a \div (-b)$.

And when either of the quantities contains more than one term, it must always be inclosed in brackets when expressed by the second method.

Thus, the quotient of $a + 2b$ by $2c = (a + 2b) \div 2c$, not $a + 2b \div 2c$, for this would mean that the quotient of $2b$ by $2c$ is to be added to a .

Rule of Signs in Division.

It is evident from the last article that—

$$\begin{aligned} (+ab) \div (+a) &= +b. \\ (-ab) \div (+a) &= -b. \\ (-ab) \div (-a) &= +b. \\ (+ab) \div (-a) &= -b. \end{aligned}$$

We have therefore the following rule :—

RULE.—In division, as in multiplication, like signs give $+$, and unlike signs give $-$.

Coefficients.

14. When a quantity can be broken up into two factors, each of those factors is called a **coefficient** of the other.

Thus, taking the quantity $4abc$, we see that 4 is the coefficient of abc , $4b$ that of ac , $4c$ that of ab , $4bc$ that of a , &c. We call 4 a *numerical* coefficient ; but when the coefficient contains a letter or letters we call it a *literal* coefficient. We often speak of the numerical coefficient of a quantity as *the coefficient* of the quantity. It is, moreover, unusual to write down unity as a numerical coefficient, and so, when an algebraical quantity has no expressed numerical coefficient, we may, conversely, consider *unity* as such.

15. *Like quantities* are generally defined as those which differ only in their *numerical* coefficients. Thus, $9x$, $3y$, $4xyz$, $12ab$ are respectively *like* to the quantities $10x$, $5y$, $6xyz$, $4ab$, while $3x^2$, $6xy$, $7b^2$ are called *unlike*.

It is sometimes convenient, however, to *consider as like quantities* those which may contain perhaps one *common* letter only, though containing other letters which are not common. Thus, for the purposes of addition and subtraction, we may consider $3ax$ and $4bx$ as *like quantities*, having $3a$ and $4b$ respectively as their coefficients. Now, their sum, since they are respectively equivalent to $3a$ and $4b$ times x , will be equivalent to $(3a + 4b)$ times x , and may then be written $(3a + 4b)x$.

Hence we learn that addition and subtraction of quantities having any common factor may, considering the uncommon factor as coefficient, be expressed according to the following rule:—

Add together the coefficients of the common factor, and take the sum as a new coefficient of the common factor.

And a similar rule will apply to the subtraction of such quantities.

Thus, we have the sum of $4axy$, $-2bcy$, $3dey$,

$$= (4ax - 2bc + 3de)y.$$

Powers.

16. We have seen (Art. 10) that abc means that a is to be multiplied by b , and the product by c . And so, if we wish to express that a is to be multiplied by a , and this product again by a , we might write the expression thus: aaa . Instead of this, however, we usually place a small figure at the top and on the right hand of the letter a , to indicate how many times the letter appears as a factor. In this case, therefore, we write a^3 . We call quantities of this form *powers* of the letter in question; and so, remembering that a may be considered as a^1 on this principle, we have:—

a , the first power of a ; a^2 , the *second* power of a ; a^3 , the *third* power of a , &c. The figure written at the right hand at the top of the letter is called the *index* or *exponent*, and it is usual to call a^2 , a^3 respectively the *square* and *cube* of a ,

from the fact they express respectively the *area* of a square whose side is a , and the *volume* of cube whose edge is a .

17. The **square root** of a quantity is that quantity which, when raised to the *second power* or *squared*, will give the original quantity.

It is generally written $\sqrt{}$. Thus, $\sqrt{16} = 4$, $\sqrt{144} = 12$.

The **cube root** of a quantity is that quantity which, when raised to the *third power* or *cubed*, will give the original quantity.

It is generally written $\sqrt[3]{}$. Thus, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[3]{1728} = 12$. And so the fourth, fifth, &c., roots are indicated by the symbols $\sqrt[4]{}$, $\sqrt[5]{}$, &c., respectively.

18. The **dimensions** of an algebraical quantity are the sum of the indices or exponents of the *literal* factors.

Thus, the dimensions of $3a^2b^3c^4 = 2 + 3 + 4 = 9$, and the dimensions of $abx^2 = 1 + 1 + 2 = 4$.

19. A **homogeneous** expression is one in which the dimensions of every term are the same.

Thus, $a^3 + 3a^2b + 3ab^2 + b^3$ is *homogeneous*,

Whereas, $5a + 3ab + 7abc$ is *not homogeneous*.

Ex. II.

If $a = 2$, $b = 3$, $c = 0$, $d = 1$,

Find the values of—

- $6a^2 + 3b^2 - 5c^2$; $ab + ac + bc$, $bc + bd + cd$.
- $a^3 + 3a^2b + 3ab^2 + b^3$; $a^3 + b^3 + c^3 - 3abc$.
- $(3a + 7b)(4a - 9b)$; $(a^3 + b^3)(a + b)(a - b)$.
- $6\left\{2a^3 - 4\left(2b^3 - \frac{2c^3 - a^3}{2}\right)\right\}$; $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$.

If $a = 1$, $b = -2$, $c = -3$, $d = 0$, $e = 4$,

Find the values of—

- $(a + b + c + d + e)^2$; $(a^3 + 2ab + b^3 - c^2) \div (a + b + c)$.
- $\frac{a^4 - d^4}{a^3 + a^2d + ad^2 + d^3}$; $\frac{c^3 + 3c^2d + 3cd^2 + d^3}{b^3 - 3b^2c + 3bc^2 - c^3}$.
- $\frac{\sqrt{c^2 + e^2} - \sqrt{5a^2 + b^2}}{ab + ac + bc}$; $\sqrt{c^2 + 3c^2 + 3c + 1}$.

8. $3(4b + 5c)^2 + 4(c + e)^2$; $abceda$.

If $x = 3$, $y = 4$, $z = 0$,

Find the value of—

9. $(3x - \sqrt{x^2 + y^2})^2 (2x + \sqrt[3]{x^2 + y^2 + 2})$.

10. $\{5x^2 + 2(y + 2)^2\} \{5x^2 - 2(y + z)^2\}$.

11. $x^2 + y^2 + z^2$.

12. $(x^2 - y^2) \div \sqrt[3]{\{3x^2 + 3(3x^2 + 3xy + y^2)y\}}$.

If $a = 7$, $b = 4$, $c = 9$, then—

13. $(a - \sqrt{c})(a^2 - b) + \sqrt{a^2 - b - c}(\sqrt{b} - \sqrt{c})^2 = 186$.

14. $a^2(c - \sqrt{b^2 - a}) + b^2(a - \sqrt{a - b^2 + c}) = 406$.

15. $\frac{3a - b}{2a - c} + \frac{5b + a}{b - c} + \frac{7c - a}{a + 2c} = \frac{6}{25}$.

16. $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = 20$.

17. $\frac{a^2b - ab^2 + b^2c - bc^2 + ac^2 - a^2c}{(a-c)(a-b)} = -5$.

18. $\frac{a + 5c}{b} - \left(\frac{a}{7} - \frac{b - 10}{c}\right) - 5a = -23\frac{2}{3}$.

If $x = 10$, $y = 11$, $z = 12$, then—

19. $(x-y)(y-z) + (y-z)(z-x) + (z-x)(x-y) = -3$.

20. $\left(\frac{x+y}{x-y} + \frac{x^2+y^2}{x^2-y^2}\right) \div \left(\frac{x-y}{x+y} - \frac{x^2-y^2}{x^2+y^2}\right) = -334\frac{1}{110}$.

21. $\{x^2 - (y + z)^2\} \cdot \frac{x + y - z}{x + y + z} = -117$.

22. $\{(x + y)^2 - (x - y)^2 - 4y^2\} \div (x - y) = 44$.

23. $\left(x + \frac{1}{y + \frac{1}{z}}\right)(y^2x^2 - 1) = 175802$.

$$24. \frac{\frac{1}{x+y} + \frac{1}{z}}{\frac{1}{x+y} - \frac{1}{z}} (z^3 - x^3 - y^3 - 2xy) = 1089.$$

$$25. \frac{y(x+z) - 2z}{xy - 1} \sqrt{\frac{z(y+1)}{y(z-1)}} = 2 \frac{2}{11}.$$

$$26. \left(\frac{2x}{y-1}\right)^2 - \left(\frac{y}{z-1}\right)^2 + \left(\frac{z}{y+1}\right)^2 = 4.$$

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

Addition.

20. RULE.—Arrange the terms of the given quantities so that *like* quantities may be under each other; add separately the positive and negative coefficients of each column; take the difference and prefix the sign of the *greater*, and annex the common letter.

(When the coefficients are all positive or all negative, we, of course, simply add them together and prefix the common sign for the coefficient of the sum.)

Ex. 1.

$$\begin{array}{r} 3a + 5b - 3c \\ 2a - 7b + 4c \\ 5a + b - 2c \\ 4a - 3b + 8c \end{array}$$

$$\text{Ans. } 14a - 4b + 7c$$

Ex. 2.

$$\begin{array}{r} -3a + 7b + c - 4d \\ 2a - 2b + 5c + 3d \\ -7a - 3b + 2c - 6d \\ 2a + 5b - 8c + 6d \end{array}$$

$$\text{Ans. } -6a + 7b - d$$

Ex. 3. Add together $5x^3 - 3y^2 + 3y$, $6y^2 + 7xy - 4x$, $4xy + 6x - 5$, $-2x^2 - 3xy + 2$.

Arranging like quantities in each expression under each other, we have:—

$$\begin{array}{r} 5x^3 \qquad \qquad - 3y^2 \qquad \qquad + 3y \\ \qquad 7xy + 6y^2 - 4x \\ \qquad 4xy \qquad \qquad + 6x \qquad \qquad - 5 \\ - 2x^2 - 3xy \qquad \qquad \qquad + 2 \\ \hline \text{Ans. } 3x^3 + 8xy + 3y^2 + 2x + 3y - 3 \end{array}$$

Ex. III.

Add together—

1. $3a - 2b, 4a + 7b, 2a + 3b, a - 5b.$
2. $9a^2 + 7b^2, -3a^2 + 4b^2, a^2 + b^2, 4a^2 - 12b^2.$
3. $a + b + c, 3a + 2b + 3c, -4a + 7b - c, 2b + 5c.$
4. $x - y - z, y - x - z, z - x - y, x + y + z.$
5. $3a^2 - 4ab + 6b^2, 7ab - a^2 - b^2, 2a^2 - 3ab - 4b^2,$
 $4a^2 + ab - b^2.$
6. $2x^4 - 7x^3 + 3, -4x^3 + 6x^2 - 2x + 7, x^4 - 2x^3 - 4x,$
 $6x^2 - 9x - 12.$
7. $2a^2 + 7ab + 3b^2 - 6a - 5b - 2, a^2 + 3a - 2b + 9,$
 $9ab - 2a - 3b + 4, -3a^2 - 12ab - 3b^2 + 5a + 10b - 15.$
8. $x^3 - x^2y - x^2z + xy^2 - xyz + xz^2, x^2y - xy^2 - xyz +$
 $y^3 - y^2z + yz^2, x^2z - xyz - xz^2 + y^2z - yz^2 + z^3.$
9. $x^4 + x^2y^2 + x^2y, -x^2y - x^2y^2 - xy^3, y^4 + xy^3 + x^2y^2.$
10. $a^3 + ab^2 + ac^2 + 2a^2b - 2a^2c - 2abc, a^2b + b^3 + bc^2 +$
 $2ab^2 - 2abc - 2b^2c, a^2c + b^2c + c^3 + 2abc - 2ac^2 - 2bc^2.$
11. $x^4 - xy^3 + xz^3 - 3x^2y + 3x^2z, 3x^2y^2 + 3x^2z^2 + 3xy^2z$
 $- 3xyz^2 - 6x^2yz, y^4 - x^2y - yz^3 + 3x^2y^2 - 3x^2yz, -3xy^3$
 $- 3xyz^2 - 3y^3z + 3y^2z^2 + 6xy^2z, z^4 + x^2z - y^2z - 3x^2yz +$
 $3x^2z^2, 3xy^2z + 3xz^3 + 3y^2z^2 - 3yz^3 - 6xyz^2.$
12. $a^4 - a^3b + 3a^2c^2 + ab^2c - 3abc^2 - b^3c, a^3b - a^2c +$
 $3abc^2 - 3ac^3 - b^2c^2 + b^3c, a^3c - a^2d + 3ac^2 - 3ac^2d + b^3c^2$
 $- b^2cd, -a^4 + a^3d - 3a^2c^2 + 3ac^2d - ab^2c + b^2cd.$

Subtraction.

21. We have seen, Art. 7 (4.), that the subtraction of a quantity is equivalent to the addition of the same quantity with its sign of affection reversed. We therefore have the following rule:—

RULE.—Change the sign of each term of the subtrahend, and proceed as in addition.

Ex. 1.

$$\begin{array}{r} 5x + 6y - 3z \\ 2x - 2y + 2z \\ \hline \text{Ans. } 3x + 8y - 5z \end{array}$$

Ex. 2.

$$\begin{array}{r} 6a^2 - 3ab + 4b^2 \\ - 2a^2 - 3ab - 2b^2 \\ \hline \text{Ans. } 8a^2 \quad \quad + 6b^2 \end{array}$$

Ex. 3.

$$\frac{2x^3 + 6xy^2 + 3y^3}{x^3 + 3x^2y + 3xy^2 + 2y^3} \div \frac{x^3 - 3x^2y + 3xy^2 - y^3}{y^3}$$

Ex. IV.

1. From $6a + 7b + 3c$ take $2a + 5b - 2c$.
2. From $2x - 3y - 8z$ take $6x - 5y - 2z$.
3. Take $5a^2 + 3ab + 4b^2 + 3a + 7b + 8$ from $6a^2 + 3b^2 - 2a$.
4. Take $6a^4 + 8a^2x^2 + x^4$ from $8a^4 + 6a^2x^2 + 2x^4$.
5. Subtract the sum of the quantities $a^4 + 2a^2b^2 + b^4$, $a^4 - 2a^2b^2 + b^4$ from $6a^4 + 8a^2b^2 + 6b^4$.
6. From $x^3 + y^3 + z^3 - 3xyz$ take $4x^3 + y^3 + 4z^3 + 3x^2z + 3xz^2 - 3xyz$.
7. From $3x^4 + 3ax^3 - 9a^2x^2 + a^3x - a^4$ take $2x^4 + 4ax^3 + 4a^2x + a^4$.
8. Take $a^3 - 5a^2b + 7ab^2 - 2b^3$ from the sum of $2a^3 - 9a^2b + 11ab^2 - 3b^3$ and $b^3 - 4ab^2 + 4a^2b - a^3$.
9. Subtract $a + b + c + d$ from $e + f + g + h$.
10. Take $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ from $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and subtract the result from their sum.
11. Add together the given quantities in the last example, and subtract the result from $3x^4 + 10x^2y^2 + 3y^4$.
12. Take $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ from $2a^2 + 2b^2 + 4ab - c^2$.

Brackets—continued.

22. It was shown, in Art. 8, that, when a quantity inclosed in brackets is to be added, we may remove the sign (+) of addition and the brackets without changing the sign of the terms within the brackets. On the contrary, when the quantity in brackets is to be subtracted, or has the sign *minus* before it, we must change the sign of every term within the brackets on removing the brackets and the sign of subtraction.

We shall now see how to simplify expressions involving brackets connected by the signs of addition and subtraction:—

Ex. 1. Simplify $(3a + 5b) - (6b - 2c) + (-2a + b - 3c)$.

The given expression $= 3a + 5b - 6b + 2c - 2a + b - 3c$, or adding together the like quantities,
 $= a - c$.

Ex. 2. Reduce to its simplest form—

$$a - (b - c) + \{b + (a - c)\} - \{(a - b) - c\}.$$

(When a pair of brackets is inclosed within another pair, it is convenient to remove the inner one first.)

Hence the given expression—

$$\begin{aligned} &= a - b + c + \{b + a - c\} - \{a - b - c\} \\ &= a - b + c + b + a - c - a + b + c = a + b + c. \end{aligned}$$

Ex. 3. Simplify the expression $\frac{3x}{4} - \frac{x-7}{2} + \frac{6x-9}{8}$.

The line separating the numerator and denominator of a fraction is a species of vinculum, since it serves to show that the *whole* numerator is to be divided by the *whole* denominator. Hence, on breaking up the two latter fractions into fractions having one term only in the numerator, we have—

$$\frac{3x}{4} - \frac{x-7}{2} + \frac{6x-9}{8} = \frac{3}{4}x - \frac{1}{2}x + \frac{7}{2} + \frac{6}{8}x - \frac{9}{8} = x + \frac{19}{8}.$$

23. As it is often necessary to inclose quantities within brackets, we shall now show how this is done.

The following rule needs no explanation:—

RULE.—When a number of terms is inclosed within brackets, if the sign placed before the brackets be +, the terms must be written down with their signs of affection *unchanged*; but, if the sign placed before the brackets be -, the sign of affection of every term placed within the brackets must be *changed*.

Thus we may express $a + b - c - d$ in any of the following ways:—

$$\begin{aligned} a + b - c - d &= a + (b - c - d) = (a + b) - (c + d) \\ &= a - (-b + c + d) = (a + b - d) - c, \text{ \&c.} \end{aligned}$$

(When the word *sign* is used in future, the student is to understand *sign of affection*, unless otherwise expressed.)

EX. V.

Simplify the expressions—

$$1. 3x - (2x - y) + (5x + 3y - 2z) - (7x - 4y - 3z).$$

$$2. (2a^3 + 2a^2b + 2ab^2) - (2a^3 + a^2b + ab^2 - b^3) + (a^3 - a^2b - ab^2 - 2b^3).$$

$$3. 1 - (1 - 2x) - \{ 3 - (4 - 3x) \} + \{ 5 - (4x - 7 - x) \}.$$

$$4. 6x^2 - (2 - 3x + x^2) + \{ -7 + (5x - 8x - 2) \} - (3 - 3 - 3x).$$

Group together the terms of the four following expressions, so that—

(i.) The first two and last two are inclosed in brackets.

(ii.) The last three are inclosed in brackets.

(iii.) The first three are so inclosed, and an inner pair of brackets used to inclose the second and third.

$$5. a - b + c - d.$$

$$6. -6a + 7b - 3c + 5d.$$

$$7. -4x^3 + 12x^2y - 12xy^2 + 4y^3.$$

$$8. a^3 - b^3 - c^3 + 3abc.$$

Add together—

$$9. ax^2 + bxy + cy^2, -dx^2 - axy + ey^2, bx^2 + 3cxy + fy^2, -2ax^2 - 2by^2.$$

$$10. ax - cy - ez, -bx + dy + fz, cx - ey - gz, -dx + fy + hz.$$

Subtract—

$$11. (2a + b)x - (3b + c)y + (4c + d)z \text{ from } (3a - d)x + (4b - a)y + (5c - b)z.$$

$$12. (y - z)a^2 + (z - x)ab + (x - y)b^2 \text{ from } (y - x)a^2 - (y - z)ab - (z - x)b^2.$$

Show that the sum of :

$$13. ax + by + cz, ay + bz + cx, az + bx + cy \text{ is } (a + b + c)(x + y + z).$$

$$14. (a + b)x + (b + c)y + (a + c)z, (b + c)x + (a + c)y + (a + b)z, \text{ and } (a + c)x + (a + b)y + (b + c)z \text{ is } 2(a + b + c)(x + y + z).$$

15. $(a - b)x + (b - c)y + (c - a)z$, $(b - c)x + (c - a)y + (a - b)z$, and $(c - a)x + (a - b)y + (b - c)z$ is zero.

16. $3(a - b) - 2(c - d)$, $4(a - c) - 3(b - d)$, $7(d - a) - 6(c - b)$ is $12(d - c)$.

17. $2(a + b) - 3(c + d)$, $3(a + c) - 4(b + d)$, $(b + c) - 3(a + d)$ is $2a - 3b - 10d$.

18. $(x - y - z)x$, $(x - y - z)y$, $(y - z)z$, $(z + x)z$ is $x^2 - y^2$.

Prove that :

$$19. 3x - 5[7x - 6x + 3\{9 - 4(2x + 2)\}] = 15 - 122x$$

$$20. (5 - 3x)\{9x - (5x - 6 - 4x)\} = 30 - 18x$$

$$21. (x - y)(z + u) + (y - z)(x + u) + (z - x)(y + u) = 0.$$

$$22. (a - b)(a + b) + (b - c)(b + c) + (c - a)(c + a) = 0.$$

$$23. 7\{x - 2x + 5\} + 9\{3 - x - 6(2 - 4x)\} = 200x - 116.$$

$$24. \frac{2x + 7}{3} + \frac{9 - 8x}{6} + \frac{5 + 6x}{9} = 4\frac{7}{18}.$$

$$25. 4x - 3[7x - \{4x + 5(6x - 3 + 5x)\}] = 10x - 45.$$

$$26. 8\{x - (4x + 3)\} + \{6x - 3(2x + 3) - 1\}(3 - 2x) = -8x - 48.$$

$$27. \{5(x + y) - 3(x - y)\}\{2x + 4y + 2\sqrt{4 - x - 2y}\} = 16x + 64y.$$

$$28. a - [2a - \{3a - (4a - 5a + 7)\}] = 3a + 7.$$

Multiplication.

24. Remembering the definition (Art. 14) of a coefficient, it follows that the product of two terms having coefficients is found by multiplying the product of the coefficients by the product of the remaining factors.

Thus, the product of $4a$ and $3b$, or $4a \times 3b = 12ab$.

Again, the terms to be multiplied may contain like letters.

Now, $a^3 = aaa$, $a^5 = aaaaa$, and hence it follows that

$$a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8.$$

And, generally, we have

$$\begin{aligned} a^m \times a^n &= aaa...to\ m\ factors \times aaa...to\ n\ factors, \\ &= aaa...to\ (m + n)\ factors = a^{m+n}. \end{aligned}$$

It therefore follows that the product of the *powers* of like letters is found by adding together the *exponents* of the like letters.

Thus, $a^4 \times a^2 = a^6$, and $a^3 \times a = a^4$, &c.

And, further, as regards the sign of the product, we have seen, in Art. 12, that *like signs* give +, and *unlike signs* give -.

There are three things, then, which must be attended to in the multiplication of algebraical terms, viz.:-

1. The *signs*.—Like signs give +, and unlike signs give -.

2. The *coefficients*.—These are to be multiplied like ordinary numbers.

3. The *letters*.—The exponents of like letters are to be added together, and the powers so obtained written side by side with the unlike letters.

Ex. 1. $\begin{array}{r} 6\ a^2b \\ 2\ a^3c \\ \hline \end{array}$ Ans. $12\ a^5bc$	Ex. 2. $\begin{array}{r} -\ 3\ xyz \\ \quad x^2yz^3 \\ \hline \end{array}$ Ans. $-3\ x^3y^2z^4$	Ex. 3. $\begin{array}{r} -\ 5\ abd \\ -\ 2\ bcd^2 \\ \hline \end{array}$ Ans. $10\ ab^2cd^2$
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25. Whenever the multiplicand or multiplier, or both, contains more than one term, it is evident the product is found by multiplying each term of the multiplicand separately by each term of the multiplier, and adding together the separate products.

Ex. 4. $\begin{array}{r} 3\ a^3 - 7\ ab + 5\ b^3 \\ 6\ ab \\ \hline \end{array}$ Ans. $18\ a^4b - 42\ a^2b^2 + 30\ ab^4$	Ex. 5. $\begin{array}{r} x^2 - 3\ x^2y + 3\ xy^2 - y^3 \\ -\ 2\ xy \\ \hline \end{array}$ Ans. $2\ x^3y - 6\ x^2y^2 - 6\ x^2y^3 + 2\ xy^4$
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Ex. 6.

$$\begin{array}{r}
 x^3 + xy + y^3 \\
 x^3 - xy + y^3 \\
 \hline
 x^4 + x^2y + x^2y^2 \\
 - x^2y - x^2y^2 - xy^3 \\
 \hline
 \text{Ans. } x^4 \qquad + x^2y^2 \qquad + y^4.
 \end{array}$$

Ex. 7.

$$\begin{array}{r}
 x^3 + (a + b)x + ab \\
 x + c \\
 \hline
 x^3 + (a + b)x^2 + abx \\
 cx^2 + (ac + bc)x + abc \\
 \hline
 x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.
 \end{array}$$

It will be observed that the terms in the above examples are all arranged according to the power of some letter. Thus, in Ex. 4, they are arranged according to the descending powers of a ; and in Exs. 5, 6, 7, they are arranged according to the descending powers of x . It matters not whether they are arranged according to ascending or descending powers, and the result would be the same if they are not so arranged. It is then, however, much easier to collect like terms, as they generally fall under each other. When we come to division we shall find it *necessary* to arrange the terms according to the power of some letter.

Ex. VI.

1. Multiply $3a + 2b$ by $4a - 3b$, and $6x + 7y$ by $3x - 5y$.

2. Multiply $x^2 + 2xy - 2y^2$ by $x - 2y$, and $15x^2 + 17xy - 4y^2$ by $2x + y$.

3. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$, and $a^3 + b^3$ by $a^2 - b^2$.

4. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$, and the product by $x^4 + y^4$.

5. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

6. Find the continued product of $x^4 + y^4$, $x^2 + y^2$, $x + y$, $x - y$.

7. Develop the expressions $(a + b)^3$, and $(x - y)^4$.
8. Multiply $5a^3 + 15a^2 + 45a + 135$ by $a^2 - 2a - 3$.
9. Multiply $1 + 3x - 7x^2$ by $x - 2$, and $a - x$ by $x^2 + ax + a^2$.
10. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4$.
11. Multiply $a + bx + cx^2 + dx^3$ by $cx + f$.
12. Find the product of $x^2 + px + q$ and $x - a$.
13. Find the continued product of $x + a$, $x + b$, $x + c$.

Show that :

14. $\left(\frac{1}{2} - \frac{a}{3} + \frac{b}{4}\right) \left(\frac{1}{2} + \frac{a}{3} + \frac{b}{4}\right) = \frac{1}{4} - \frac{a^2}{9} + \frac{b}{4} + \frac{b^2}{16}$.
15. $\left(4x^2 - \frac{2}{5}x + \frac{1}{25}\right) \left(2x + \frac{1}{5}\right) = 8x^2 + \frac{1}{125}$.
16. $(x^4 + ax^3 + a^2x^2 + a^3x + a^4)(x^5 + a^5)(x - a) = x^{10} - a^{10}$.
17. $(x^2 + ax + a^2)(x^2 - ax + a^2)(x + a)(x - a) = x^8 - a^8$.
18. $(x + 1)(x + 3)(x + 4)(x - 8) = x^4 - 45x^2 - 140x - 96$.
19. $(3a^m + 4b^n)(4a^m - 3b^n) = 12a^{2m} + 7a^mb^n - 12b^{2n}$.
20. $(x^3 + 2ax^2 + 2a^2x + a^3)(x^3 - 2ax^2 + 2a^2x - a^3) = x^6 - a^6$.
21. $(b - c)(a^2 + ma + n) + (c - a)(b^2 + mb + n) + (a - b)(c^2 + mc + n) = ab(a - b) + bc(b - c) + ac(c - a)$.
22. $(a^mx^n + b^px^q)(a^m - b^px^{q-n}) = a^{2m}x^n - b^{2p}x^{2q-n}$.
23. $(a + b + c + d)(a + b - c - d)(a - b - c + d)(a - b + c - d) + (b + c + d - a)(a + c + d - b)(a + b + d - c)(a + b + c - d) = 16abcd$.
24. $(a + b + c)(a + b - c) + (b + c - a)(c + a - b) = 4ab$.
25. $\{a^{(x-1)y} - b^{(y-1)x}\}(a^y + b^x) = a^{xy} - b^{xy} + a^{(x-1)y}b^x - a^yb^{(y-1)x}$.
26. $\left(\frac{1}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{3}} - \frac{1}{4}c^{\frac{1}{4}}\right)\left(\frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}b^{\frac{1}{3}} + \frac{1}{4}c^{\frac{1}{4}}\right) = \frac{1}{4}a - \frac{1}{9}b^{\frac{1}{3}} - \frac{1}{16}c^{\frac{1}{4}} + \frac{1}{6}b^{\frac{1}{3}}c^{\frac{1}{6}}$.

$$27. (3a + 6b^{\frac{1}{2}} - 9c^{\frac{1}{2}} + 12d^{\frac{1}{2}})(2a + 4b^{\frac{1}{2}} + 6c^{\frac{1}{2}} - 8d^{\frac{1}{2}}) \\ = 6(a^2 + 4b - 9c^{\frac{1}{2}} - 16d^{\frac{1}{2}} + 4ab^{\frac{1}{2}} + 24cd^{\frac{1}{2}}).$$

$$28. (x^2 - y^2)(x + y) - (x^2 + y^2)(x - y) = 2x^2y - 2xy^2.$$

$$29. (2x - 3y)^2 + (4y - 5x)^2 + (3x - y)^2 - 3(2x - 3y) \\ (4y - 5x)(3x - y) = 0.$$

$$30. (x + y)^4 + (x - y)^4 - 2(x^2 - y^2)^2 = 16x^2y^2.$$

23. We have explained in the last Article the general method to be pursued in Multiplication. There are, however, many algebraical expressions which may be multiplied by inspection.

I. Expressions of the form $(a + b)^2$.

It is easily found by long multiplication that

$$(a + b)^2 = a^2 + 2ab + b^2,$$

and this, expressed in ordinary language, may be read thus :—

The square of the sum of two quantities is the square of the first, plus twice their product, plus the square of the second.

(This rule is evidently true whether the quantities are *positive* or *negative*.)

$$\text{Ex. 1. } (3a + 7b)^2 = (3a)^2 + 2(3a)(7b) + (7b)^2 = 9a^2 \\ + 42ab + 49b^2.$$

$$\text{Ex. 2. } (6x - 5)^2 = (6x)^2 + 2(6x)(-5) + (-5)^2 = \\ 36x^2 - 60x + 25.$$

$$\text{Ex. 3. } (a + b + c)^2 = \{(a + b) + c\}^2 \\ = (a + b)^2 + 2(a + b)c + c^2 \\ = (a^2 + 2ab + b^2) + 2(ac + bc) + c^2 \\ = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

REMARK.—It is a very common mistake with beginners to write down $a^2 + b^2$ as the square of $(a + b)$, thereby leaving out *twice the product of the quantities a and b*. They should impress this formula, viz. :—

$$(a + b)^2 = a^2 + 2ab + b^2.$$

thoroughly upon the mind.

II. The form $(a + b)(a - b)$.

It may be easily found by long multiplication that

$$(a + b)(a - b) = a^2 - b^2,$$

which, in ordinary language, may be thus expressed :—

The product of the sum and difference of two quantities is the difference of the squares of the quantities.

This formula may be applied in all cases where the terms of the quantities to be multiplied are of the same magnitude, but when *some* of them differ in sign.

Ex. 1. $(a + b + c + d)(a + b - c - d)$

$$\begin{aligned} &= \{ (a + b) + (c + d) \} \{ (a + b) - (c + d) \} \\ &= (a + b)^2 - (c + d)^2 = (a^2 + 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= a^2 + b^2 - c^2 - d^2 + 2ab - 2cd. \end{aligned}$$

Ex. 2. $(a^3 - 3a^2b + 3ab^2 - b^3)(a^3 + 3a^2b + 3ab^2 + b^3)$

$$\begin{aligned} &= \{ (a^3 + 3ab^2) - (3a^2b + b^3) \} \{ (a^3 + 3ab^2) + (3a^2b + b^3) \} \\ &= (a^3 + 3ab^2)^2 - (3a^2b + b^3)^2 \\ &= (a^6 + 6a^4b^2 + 9a^2b^4) - (9a^4b^2 + 6a^2b^4 + b^6) \\ &= a^6 - 3a^4b^2 + 3a^2b^4 - b^6. \end{aligned}$$

The principle to be adopted in all such cases is to find what terms in the given quantities are *exactly alike*, and put them *first* in brackets, when the remaining terms will fall into the proper form. Thus—

$$\begin{aligned} &(3a + 7b + 5c - d)(3a - 7b + 5c + d) = \{ (3a + 5c) \\ &+ (7b - d) \} \{ (3a + 5c) - (7b - d) \}. \end{aligned}$$

$$\begin{aligned} &\text{And } (3a - 7b - 5c + d)(3a + 7b + 5c + d) = \{ (3a + d) \\ &- (7b + 5c) \} \{ (3a + d) + (7b + 5c) \}. \end{aligned}$$

(III.) The form $(x + a)(x + b)$.

By multiplication we find that—

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

which in ordinary language may be thus expressed :—

The product of two binomials containing a common term and an uncommon term, is the square of the common term, plus the sum of the uncommon terms multiplied into the common term, plus the product of the uncommon term.

Thus—

$$\begin{aligned}(x+2)(x+3) &= x^2 + (2+3)x + 2 \times 3 = x^2 + 5x + 6. \\ (x+6)(x-1) &= x^2 + (6-1)x + 6(-1) = x^2 + 5x - 6. \\ (x-5)(x-7) &= x^2 - (5+7)x + (-5)(-7) = x^2 - 12x \\ &+ 35. \\ (x+3a)(x-5a) &= x^2 + (3a-5a)x + (3a)(-5a) = x^2 \\ &- 2ax - 15a^2.\end{aligned}$$

And so,

$$\begin{aligned}(x+a+b)(x+c+d) &= (x+\overline{a+b})(x+\overline{c+d}) \\ &= x^2 + (\overline{a+b+c+d})x + (\overline{a+b})(\overline{c+d}).\end{aligned}$$

We may extend this formula to any number of factors.

Thus, by multiplication, we find that—

$$\begin{aligned}(x+a)(x+b)(x+c) &= x^3 + (a+b+c)x^2 + (ab+ac \\ &+ bc)x + abc, \\ (x+a)(x+b)(x+c)(x+d) &= x^4 + (a+b+c+d) \\ &x^3 + (ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd \\ &+ bcd)x + abcd.\end{aligned}$$

Law of Formation of the Terms.

1. It will be seen that the coefficient of the first term is in each case *unity*; that of the second term, the sum of the uncommon letters taken *singly*; that of the third term, the sum of the uncommon letters taken *two together*; that of the fourth term, the sum of the uncommon letters taken *three together*, &c.

2. The power of the common letter is in the first term that of the number of the binomials, and it sinks *one* every term.

$$\text{Ex. } (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1 \times 2 + 1 \times 3 + 2 \times 3)x + 1 \times 2 \times 3 = x^3 + 6x^2 + 11x + 6.$$

(IV.) The form $(a+b+c+d+\&c.)^2$.

By ordinary multiplication or otherwise we have $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$; and a similar result follows if we take a larger number of terms.

We may hence deduce the following rule:—

The square of the sum of any number of quantities is the sum of the squares of the quantities together with twice the sum of the products formed by multiplying the first quantity

into all that follow separately, then the second into all that follow, the third into all that follow, &c.

$$\begin{aligned} \text{Ex. 1. } (1 + 2x + 3x^2)^3 &= 1 + 4x^3 + 9x^4 \\ &\quad + 4x + 6x^2 + 12x^3 \\ &= 1 + 4x + 10x^2 + 12x^3 + 9x^4. \end{aligned}$$

$$\begin{aligned} \text{Ex. 2. } (a^3 + 3a^2b + 3ab^2 + b^3)^3 &= a^9 + 9a^6b^3 + 9a^2b^4 \\ &\quad + 6a^5b + 6a^4b^2 + 2a^3b^3 + b^6 \\ &\quad + 18a^3b^3 + 6a^2b^4 + 6ab^5 \\ &= a^9 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6. \end{aligned}$$

Ex. VII.

1. Find by inspection the squares of $x - y$, $3a - 5b$, $4c^2 + d^2$, $3x^2 - 2y^2$.

2. Find the continued product of $a + b$, $a^2 + b^2$, $a^4 + b^4$, and $a - b$.

3. Multiply $mx + ny$ by $mx - ny$, and $5a^2 - 3b^2$ by $10a^2 + 6b^2$.

4. Find the value of $(a + b + c + d)(a - b + c - d)$, and of $(a + b - c - d)(a + b + c + d)$.

5. Show that $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2) = x^4 - 2x^2y^2 + y^4$.

6. Multiply $x + 5$ separately by $x - 1$, $x + 2$, $x - 3$, $x - 5$.

7. What is the continued product of $x - 2$, $x + 3$, $x - 6$, $x + 5$?

8. If $2s = a + b + c$, find the value of $s(s - a)(s - b)(s - c)$.

9. If $2s = a + b + c + d$, what is the value of—

$$(2s - 2a)^2 + (2s - 2b)^2 + (2s - 2c)^2 + (2s - 2d)^2?$$

Prove that—

$$10. (ax + by)^2 + (cx + dy)^2 + (ay - bx)^2 + (cy - dx)^2 = a^2 + b^2 + c^2 + d^2 (x^2 + y^2).$$

$$11. \{(ac - bd)x + (ad + bc)y\}^2 + \{(ac + bd)y - (ad + bc)x\}^2 = (a^2 + b^2)(c^2 + d^2)(x^2 + y^2).$$

$$12. (ax + by + cz)^2 = (a + b + c)(ax^2 + by^2 + cz^2) - ab(x - y)^2 - ac(x - z)^2 - bc(y - z)^2.$$

13. $(x + a)(x + b)(x + c) = (x - a)(x - b)(x - c) + 2 \{ (a + b + c)x^2 + abc \}.$
14. $(m + n + p + q)^2 = (m + n)^2 + (m + p)^2 + (m + q)^2 + (n + p)^2 + (n + q)^2 + (p + q)^2 - 2(m^2 + n^2 + p^2 + q^2).$
15. $(a^2 + b^2 + c^2)(m^2 + n^2 + p^2 + q^2) = (am + bn + cp)^2 + (an - bm + cq)^2 + (ap - bq - cm)^2 + (aq + bp - cn)^2.$
16. $(a - b)(b - c) + (b - c)(c - a) + (c - a)(a - b) = 3(ab + ac + bc) - (a + b + c)^2.$
17. $(x + y + z + a)^2 - (x - y - z + a)^2 = 4(x + a)(y + z).$
18. $\{(a - b)^2 + 4ab\} \{(a + b)^2 - 4ab\} \{a^4 - b^4 + 2ab(a^2 - b^2)\} = (a + b)^6(a - b)^2.$
19. $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2.$
20. $4(a^2 + b^2)(c^2 + d^2) = (a + b)^2(c + d)^2 + (a - b)^2(c - d)^2 + (a + b)^2(c - d)^2 + (a - b)^2(c + d)^2.$
21. $8(a^4 + b^4) = (a + b)^4 + 6(a^2 - b^2)^2 + (a - b)^4.$
22. $(x - y)^2 + (y - z)^2 + (z - x)^2 + 2(x - y)(y - z) + 2(x - y)(z - x) + 2(y - z)(z - x) = 0.$
23. $(a + b + c)(b + c - a)(a + c - b)(a + b - c) = 4a^2b^2$, when $a^2 + b^2 = c^2.$
24. $2 \{ (ax^m + by^n)^2 + (ay^n - bx^m)^2 \} = (a^2 + b^2) \{ (x^m + y^n)^2 + (x^m - y^n)^2 \}.$

Division.

27. As in multiplication, we have especially three things to attend to, viz. :—

1. The signs.

We have learnt (Art. 13) that like signs give +, and unlike signs give -.

2. The coefficients.

Understanding here the *numerical* coefficients, it is plain that they may be divided as ordinary arithmetical quantities.

3. The letters.

As the product of the quantities a and b is expressed by ab , it follows that the quotient of ab by either of the factors a or b will give the remaining factor b or a respectively.

Thus, $ab \div a$ or $\frac{ab}{a} = b$, and $ab \div b$ or $\frac{ab}{b} = a$.

And so, $xyz \div xz = y$, and $pqr \div qr = pr$.

We may then conclude that *when the divisor is contained as a factor in the dividend, the quotient is found by omitting from the dividend those of its factors which constitute the divisor.*

If the divisor be *not* contained as an exact factor in the dividend, we may then express the quotient symbolically.

Thus, $xy \div ab = \frac{xy}{ab}$.

When, however, the dividend and divisor have a common factor, it is plain that we may, as in arithmetic, strike out of the numerator and denominator of the symbolical quotient this common factor.

Thus, $5abc \div bd = \frac{5abc}{bd} = \frac{5ac}{d}$.

And $16xyz \div 10axz = \frac{16xyz}{10axz} = \frac{8y}{5a}$.

28. A power of a quantity is divided by any other power of the same quantity by subtracting the index of the divisor from that of the dividend, the quotient being that power of the quantity whose index is the remainder so obtained.

1. Let the power of the quantity in the dividend be the *higher*.

We have $a^5 = aaaaa$, and $a^3 = aaa$.

$$\therefore a^5 \div a^3 = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Or, generally, m being greater than n , since

$a^m = aaa \dots$ to m factors, and $a^n = aaa \dots$ to n factors,

we have—

$$a^m \div a^n = \frac{aaa \dots \text{to } m \text{ factors}}{aaa \dots \text{to } n \text{ factors}} = aaa \dots (m - n) \text{ factors} = a^{m-n}.$$

2. Let the power of the quantity in the dividend be the *lower*.

Suppose we have to divide a^4 by a^7 .

$$\text{We have } a^4 \div a^7 = \frac{aaaa}{aaaaaaa} = \frac{1}{aaa} = \frac{1}{a^3}.$$

We may, however, so express the result that it shall agree exactly with the proposition at the head of this article.

For we may conceive of a^3 as representing the *product* of *unity* and the quantity a^3 . We shall therefore be perfectly consistent if we allow a^3 to represent the *quotient* of *unity* by the quantity a^3 .

We shall then have $a^{-3} = \frac{1}{a^3}$, and hence we get from the above result—

$$a^4 \div a^7 = a^{-3} = a^{4-7}.$$

Or, generally, m being less than n , we have—

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\text{aaa.....to } m \text{ factors}}{\text{aaa.....to } n \text{ factors}} = \frac{1}{\text{aaa.....}(n - m) \text{ factors}} \\ &= \frac{1}{a^{n-m}}; \text{ or, using the notation just explained,} \\ &= a^{-(n-m)} = a^{-n+m} = a^{m-n}. \end{aligned}$$

3. Let the powers of the quantities in the dividend and divisor be *equal*.

It is evident that their quotient is *unity*.

$$\text{Thus, } \frac{a^4}{a^4} = \frac{aaaa}{aaaa} = \frac{1}{1} = 1.$$

And so, $\frac{a^m}{a^m} = 1$.

If, however, we *assume* the principle proved in the two cases above to hold here, we have—

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

It follows therefore that $a^0 = 1$.

COR.—From the above interpretation of negative indices it follows that the same rules for multiplication and division of quantities involving them may be applied as in the case of positive indices.

$$\text{Thus, } a^5 \times a^{-3} = a^5 \times \frac{1}{a^3} = \frac{a^5}{a^3} = a^{5-3}.$$

$$\text{And so, } a^4 \div a^{-3} = a^{4-(-3)} = a^7.$$

$$\text{Ex. 1. } \frac{20 x^2 y z}{4 x z} = 5 x y.$$

$$\text{Ex. 2. } \frac{-45 a^3 b^3 c^4}{15 a b^3 c^3} = -3 a^2 c.$$

$$\text{Ex. 3. } \frac{-7 a^3 + 21 a^2 b - 14 a b^2}{-7 a} = a^2 - 3 a b + 2 b^2.$$

$$\begin{aligned} \text{Ex. 4. } & \frac{4 x^3 y z^3 - 6 x^2 y^2 z^4 + 42 x y^4 z^4}{3 x^2 y^4 z^3} \\ &= \frac{4 x^2}{3 y^2 z^3} - \frac{2}{y z} + \frac{14}{x z}, \text{ or} \\ &= \frac{4}{3} x^2 y^{-2} z^{-3} - 2 y^{-1} z^{-1} + 14 x^{-1} z^{-1}. \end{aligned}$$

$$\text{Ex. 5. Divide } 3 a^3 + 13 a b - 10 b^3 \text{ by } a + 5 b.$$

When the divisor, as in this example, contains more than one term, it is generally convenient to follow the method of arithmetical long division. Thus—

$$\begin{array}{r} a + 5 b \overline{) 3 a^3 + 13 a b - 10 b^3} \\ \underline{3 a^3 + 15 a b} \\ - 2 a b - 10 b^3 \\ \underline{- 2 a b - 10 b^3} \end{array}$$

$$\text{Ex. 6. Divide } a^4 + a^2 b^2 + b^4 \text{ by } a^2 + a b + b^2.$$

$$\begin{array}{r} a^2 + a b + b^2 \overline{) a^4 + a^2 b^2 + b^4} \\ \underline{a^4 + a^3 b + a^2 b^2} \\ - a^3 b - a^2 b^3 - a b^4 \\ \underline{- a^3 b - a^2 b^3 - a b^4} \\ a^2 b^3 + a b^4 + b^4 \\ \underline{a^2 b^3 + a b^4 + b^4} \end{array}$$

It will be seen that in the last two examples care has been taken to keep the terms of the divisor, dividend, and successive remainders arranged according to the ascending or descending powers of some letter. In these cases we have arranged the terms according to the descending powers of a , and, as therefore follows, according to the ascending powers of b . Want of care in this respect will often render the operation of finding the true quotient *tiresome*, if not impossible. The next two examples will illustrate this point. They may be

attempted first by the student, keeping the terms in the order as given.

Ex. 7. Divide $2 - 7x - 15x^2$ by $5x - 1$.

$$\begin{array}{r} -1 + 5x \overline{) 2 - 7x - 15x^2} \\ \underline{2 - 10x} \\ 3x - 15x^2 \\ \underline{3x - 15x^2} \\ 0 \end{array}$$

$$\begin{array}{r} \text{Or thus, } 5x - 1 \overline{) -15x^2 - 7x + 2} \\ \underline{-15x^2 + 3x} \\ -10x + 2 \\ \underline{-10x + 2} \\ 0 \end{array}$$

Ex. 8. Divide $x^3 + y^3$ by $x^{-1} + y^{-1}$.

Here we have the powers of x in the dividend descending, while in the divisor they are ascending. Arranging them in the divisor as in the dividend, the operation is easy. Thus—

$$\begin{array}{r} y^{-1} + x^{-1} \overline{) x^3 + y^3} \\ \underline{x^3 + x^2y} \\ -x^2y + y^3 \\ \underline{-x^2y - xy^2} \\ xy^2 + y^3 \\ \underline{xy^2 + y^3} \\ 0 \end{array}$$

We shall work the next example in two ways to illustrate, firstly, the above point again ; and, secondly, to show how the operation may be sometimes abbreviated by the use of brackets.

Ex. 9. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

$$\begin{array}{r} x + y + z \overline{) x^3 + y^3 + z^3 - 3xyz} \\ \underline{x^3 + x^2y + x^2z} \\ -x^2y - x^2z - 3xyz \\ \underline{-x^2y - xy^2 - xyz} \\ -x^2z + xy^2 - 2xyz \\ \underline{-x^2z - xyz - xz^2} \\ xy^2 - xyz + xz^2 + y^3 \\ \underline{xy^2 + y^3 + y^2z} \\ -xyz + xz^2 - y^2z \\ \underline{-xyz - y^2z - yz^2} \\ xz^2 + yz^2 + z^3 \\ \underline{xz^2 + yz^2 + z^3} \\ 0 \end{array}$$

Or thus, inclosing the last two terms of the divisor in brackets—

$$\begin{array}{r}
 x + (y + z) \big) x^3 - 3xyz + y^3 + z^3 \big(x^2 - (y + z)x + (y^2 - yz + z^2) \\
 \underline{x^2 + (y + z)x^2} \\
 \quad - (y + z)x^2 - 3xyz \\
 \quad \underline{-(y + z)x^2 - (y^2 + 2yz + z^2)x} \\
 \qquad (y^2 - yz + z^2)x + y^3 + z^3 \\
 \qquad \underline{(y^2 - yz + z^2)x + y^3 + z^3}
 \end{array}$$

It will be seen in both the above operations that we have brought down the terms of the dividend only when the subtrahends indicated they were required. This often prevents much useless repetition.

Ex. VIII.

Find the quotient of—

1. $28 a^2b - 7 ab^2 + 14 b^3$ by $7 b$; $3 x^2y^3 - 12 xy^3$ by $3 xy$.

2. $-6 a^3b + 15 a^4b^2 - 20 a^5b^3$ by $-3 ab$; $4 x^3y^3 + 6 x^2y^3 + 4 xy^4$ by $2 xy^2$.

3. $ax^{m+n} + bx^my^n + cy^{2n}$ by x^{m+n} ; $ax^{2m-2n} + bx^{m-n}y^n + cy^{2n}$ by x^{m-n} .

4. $30 x^3 + 2 xy - 12 y^2$ by $5 x - 3 y$; and by $6 x + 4 y$.

5. $1 + 2 x + 3 x^2 + 2 x^3 + x^4$ by $1 + x + x^2$.

6. $12 - 19 x - 21 x^2$ by $7 x - 3$; and by $3 x + 4$.

7. $x^4 - 4 x^2y^2 + 12 xy^3 - 9 y^4$ by $x - 3 y$; and by $x + y$.

8. $x^5 - y^5$ by $x - y$; and $x^5 + y^5$ by $x + y$.

9. $acx^{m+n} + adx^my^n + bcx^ny^n + bdy^{m+n}$ by $cx^n + dy^n$.

10. $a^5 - a^4b^2 - a^2b^4 + b^6$ by $a^3 - a^2b - ab^2 + b^3$.

11. $b^3 + ab + bc + ac$ by $a + b$; and $a^2 + ab + bc + ac$ by $a + c$.

12. $a + (a + b)x + (a + b + c)x^2 + (a + b + c)x^3 + (b + c)x^4 + cx^5$ by $1 + x + x^2 + x^3$.

13. $a^5 - pa^4 + qa^3 - qa^2 + pa - 1$ by $a - 1$.

14. $a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd$ by $a - b + c + d$; and that of this quotient by $a - b - c - d$.

15. Show that the remainder, after the division of $x^4 - px^3 + qx^2 - rx + s$ by $x - a$, is $a^4 - pa^3 + qa^2 - ra + s$.

16. Divide $x^4 - y^4$ by $x^{-2} - y^{-2}$, and $x^2 + x^{-2} + 2$ by $x + x^{-1}$.

17. Show that the quotient of 1 by $1 + x$, is $1 - x + x^2 - x^3 + \&c.$ *ad infinitum*.

18. Show that the quotient of 1 by $1 - 3x + 3x^2 - x^3$ is $1 + 3x + 6x^2 + 10x^3 + 15x^4 + \&c.$ *ad infinitum*.

19. Divide $(x + y)^4 - 2(x + y)^2 z^2 + z^4$ by $x + y - z$.

20. Divide $(a - c)^3 - 3(a - c)^2(b - d) + 3(a - c)(b - d)^2 - (b - d)^3$ by $a - b - c + d$.

Prove that :

$$21. \{ab(b - a) + ac(a - c) + bc(c - b)\} \div (b - c) = (a - b)(c - a).$$

$$22. \{a^3(b^2 - c^2) - ab(2b^2 + bc - c^2) + b^3(b + c)\} \div (ab - ac - b^2) = (a - b)(b + c).$$

$$23. \frac{ab + 2a^2 - 3b^2 - 4bc - ac - c^2}{2a + 3b + c} = a - b - c.$$

$$24. \frac{12a^2 + 7ab - 12b^2 + ac - 7bc - c^2}{3a + 4b + c} = 4a - 3b - c.$$

$$25. \frac{x^2 - 2ax^2 + (a^2 - ab - b^2)x + a^2b + ab^2}{(x - a)(x + b)} = x - a - b.$$

$$26. \frac{(x^2 - 1)(y^2 - 1)(z^2 - 1) + (x + yz)(y + xz)(z + xy)}{x^2 + y^2 + z^2 + 2xyz - 1} = xyz + 1.$$

$$27. \frac{x^2(z - z) + y^2(z - x) + z^2(x - y)}{(x - y)(y - z)(x - z)} = x + y + z.$$

$$28. \frac{x^4(y - z) + y^4(z - x) + z^4(x - y)}{(x - y)(y - z)(x - z)} = x^2 + y^2 + z^2 + xy + yz + xz.$$

$$29. \frac{abcd(a+b+c+d)^2 - (abc+bcd+cda+dab)^2}{(bc-ad)(ca-bd)} = cd - ab.$$

$$30. \frac{(a+b)^2 + (a+c)^2 + (b+c)^2 - 3(a+b)(a+c)(b+c)}{a^2 + b^2 + c^2 - ab - ac - bc} = 2(a+b+c).$$

$$31. \frac{(x-y)^2 + (y-z)^2}{x-z} = (x-y)^2 + (y-z)^2 - (x-y)(y-z).$$

$$32. \frac{x^m - 1}{x^n - 1} = x^{(n-1)m} + x^{(n-2)m} + \dots + x^2 + x + 1.$$

$$33. \frac{a^4 - b^4}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} = a^{\frac{3}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{3}{2}} + \dots + a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{3}{2}}.$$

$$34. (x-y) + (\sqrt{x} - \sqrt{y}) = x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}.$$

Factors.

29. The *ordinary* method of finding the quotient of two algebraical quantities having been explained in the last article, we shall now proceed to show how, in certain cases, this method may be avoided, and the quotient written down at sight. It may be remarked at the outset, that the resolution of algebraical expressions into their elementary factors is a subject of very great importance, and one which the student will do well to thoroughly master.

(I.) The form $x^2 + 2ax + a^2$.

We have seen (Art. 26) that $x^2 + 2ax + a^2 = (x+a)^2$. Hence the *sum of the squares* of two quantities, together with twice the product of the quantities, is equal to the *square of the sum* of the quantities.

And hence, any algebraical expression, which can be thrown into the form $(x^2 + 2ax + a^2)$, is of necessity a perfect square.

Thus—

$$x^2 + 6x + 9 = x^2 + 2(3)x + 3^2 = (x+3)^2.$$

$$a^2 - 10ab + 25b^2 = a^2 + 2a(-5b) + (-5b)^2 = (a-5b)^2.$$

$$16a^2x^2 - 56abxy + 49b^2y^2 = (4ax)^2 + 2(4ax)(-7by) + (-7by)^2 = (4ax - 7by)^2.$$

$$3 ab^2x^4 - 6 abcxxy + 3 ac^2y^2 = 3a(b^2x^2 - 2bcxy + c^2y^2) = 3a\{(bx)^2 - 2(bx)(cy) + (cy)^2\} = 3a(bx - cy)^2.$$

(II.) The form $a^2 - b^2$.

We have seen (Art. 26, II.) that $a^2 - b^2 = (a + b)(a - b)$.

Hence the *difference of the squares* of two quantities is equal to the product of the sum and difference of the quantities. Thus—

$$\begin{aligned} a^2x^2 - b^2y^2 &= (ax)^2 - (by)^2 = (ax + by)(ax - by). \\ x^4 - y^4 &= (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) \\ &= (x^2 + y^2)(x + y)(x - y). \\ a^2 + b^2 - c^2 - d^2 + 2ab - 2cd &= (a^2 + 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= (a + b)^2 - (c + d)^2 \\ &= \{(a + b) + (c + d)\} \{(a + b) - (c + d)\} \\ &= (a + b + c + d)(a + b - c - d). \\ x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= \{(x^2 + y^2) + xy\} \{(x^2 + y^2) - xy\} \\ &= (x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

(III.) The form $x^2 + px + q$.

This form evidently includes both the preceding, for the first form—viz., $x^2 + 2ax + a^2$ is included, since q may be the square of half p ; and the second is included—viz., $x^2 - a^2$, since we may have $p = 0$, and q a negative square quantity.

Now, the resolution into elementary rational factors of the quantity $x^2 + px + q$ is not always possible; but, since (Art. 26, III.),

$$x^2 + (a + b)x + ab = (x + a)(x + b),$$

we have the following rule, when the quantity admits of resolution.

RULE.—If the third term q of the quantity $x^2 + px + q$ can be broken up into two factors, a and b , such that the sum of these will give the coefficient of x , then the elementary factors of $x^2 + px + q$ are $x + a$ and $x + b$.

Thus, $x^2 + 7x + 12 = (x + 3)(x + 4)$; for the *product* of 3 and 4 is 12, and their *sum* is 7, the coefficient of x .

$x^2 - x - 30 = (x - 6)(x + 5)$; for the *product* of -6 and 5 is -30 , and their *sum* is -1 .

And so, $x^2 - 18x + 32 = (x - 2)(x - 16)$,

And $a^2 + 3ab - 108b^2 = a^2 + (12b - 9b)a + (12b)(-9b) = (a + 12b)(a - 9b)$.

(IV.) The form $ax^2 + bx + c$.

This is the *general* form of a trinomial. The following remarks, though equally applying to each of the three preceding forms, are especially intended to be *practically* applied to trinomials not included by them.

The above form will include such expressions as the following:— $20x^2 + 11x - 42$, $6x^2 - 37x + 55$.

It is evident that the product of the first terms of the factors will be the first term of the given trinomial, and that the product of the last terms of the factors will be the third term of the given trinomial.

And, further, when the third term is *negative*, the last term of one factor must have the sign $+$, and the last term of the other the sign $-$; but, when the third term is *positive*, the last terms of the factors must have the same sign as the middle term.

Thus, $12x^2 - 31x - 30 = (4x + 3)(3x - 10)$.

Here the factors of $12x^2$ are either $3x$ and $4x$, $6x$ and $2x$, $12x$ and x , and the factors of 30 either 5 and 6 , 3 and 10 , 2 and 15 , 1 and 30 ; and we must give a $+$ sign to one of each of these latter pairs, and a $-$ sign to the other. It is easily found on trial that, in order to obtain $-31x$ as the middle term, the factors of the trinomial must be $4x + 3$ and $3x - 10$.

So we have $10a^2 - 41ab + 21b^2 = (5a - 3b)(2a - 7b)$, and $acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$.

(V.) The forms $x^n + y^n$ and $x^n - y^n$.

We shall show in the next article that a rational integral algebraical expression, involving x , contains $x - a$ as a factor when it vanishes on substituting a for x .

Hence, $x^n + y^n$ and $x^n - y^n$ must each vanish on putting y for x , if they contain $x - y$ as a factor, n being an integer.

The former becomes $y^n + y^n$ or $2y^n$, and the latter $y^n - y^n$ or 0 . We therefore conclude that—

$x^n + y^n$ does not contain $x - y$ as a factor, and that—

$x^n - y^n$ does contain $x - y$ as a factor, whether n be even or odd.

Again, on the same principle, they must each vanish if they contain $x + y$ as a factor, on putting $-y$ for x .

The former becomes $(-y)^n + y^n$, which vanishes when n is odd, and the latter becomes $(-y)^n - y^n$, which vanishes when n is even.

Hence we conclude that—

$x^n + y^n$ contains $x + y$ as a factor when n is odd, and

$x^n - y^n$ contains $x + y$ as a factor when n is even.

Now, the quotient of either of these quantities by $x + y$ or $x - y$ can in any particular case be found by long division.

We thus find that—

$$x^5 + y^5 = (x + y) (x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$

$$x^4 - y^4 = (x - y) (x^3 + x^2y + xy^2 + y^3),$$

$$x^3 - y^3 = (x - y) (x^2 + xy + y^2).$$

The law of formation of the co-factor in each case is easy to see; and if we may *assume* this apparent law as generally true, we may conclude that, when an algebraical quantity is of the form $x^n + y^n$ or $x^n - y^n$, and it contains $x + y$ or $x - y$ as a factor, the law of formation of the co-factor is as follows:—

Law of Formation of Co-Factor.

1. The terms are homogeneous, and of dimensions one degree lower than the given expression, the power of x in the first term being $n - 1$, and diminishing each successive term by *unity*; and the power of y increasing each successive term by *unity*, and first appearing in the second term.

2. The coefficient of every term is *unity*.

3. The *signs* are alternately $+$ and $-$, when $x + y$ is the corresponding elementary factor; and are all $+$, when $x - y$ is the corresponding elementary factor.

Ex. 1. $x^5 + 32 = x^5 + 2^5 = (x + 2) (x^4 + x^3 \cdot 2 + x^2 \cdot 2^2 + x \cdot 2^3 + 2^4) = (x + 2) (x^4 + 2x^3 + 4x^2 + 8x + 16).$

Ex. 2. $a^5 - b^5 = (a^5)^2 - (b^5)^2 = (a^3 + b^3) (a^2 - b^2) = (a + b) (a^2 - ab + b^2) \cdot (a - b) (a^2 + ab + b^2) = (a + b) (a - b) (a^3 - ab + b^2) (a^2 + ab + b^2).$

30. The remainder of the division of a rational integral function of x by $x - a$ may be found by putting a for x in the given function.

DEF.—A function of x is an algebraical expression involving x ; and a rational integral function of x is an expression of the form $ax^n + bx^{n-1} + \&c. + sx + t$, where all the powers of x are integral and positive.

Let $f(x)^*$ be a rational integral function of x , and suppose Q to be the quotient, and R the remainder on dividing the function by $x - a$.

Then, evidently—

$$Q(x - a) + R = f(x) \text{ identically.}$$

And this identity must hold for all values of x , and therefore holds when $x = a$.

$$\begin{aligned} \text{In this case we have } Q(a - a) + R &= f(a) \\ 0 + R &= f(a) \\ \text{or } R &= f(a). \end{aligned}$$

Now $f(a)$ is the result of putting a for x in the given function, and is, as we have just shown, the remainder on dividing the given function by $x - a$.

COR. 1. When there is no remainder, we must, of course, have $f(a) = 0$. Hence, a given rational integral function of x vanishes when a is put for x , if it be divisible by $x - a$.

EX. 1. The remainder, after the division of $2x^3 - 5x^2 + 6x + 7$ by $x - 2$ is 15.

For, putting $x = 2$, we have—

$$2x^3 - 5x^2 + 6x + 7 = 2 \cdot 2^3 - 5 \cdot 2^2 + 6 \cdot 2 + 7 = 15.$$

EX. 2. The function—

$$x^3 - 2x^2 + 5x - 52 \text{ is divisible by } x - 4.$$

For, putting $x = 4$, we have—

$$x^3 - 2x^2 + 5x - 52 = 4^3 - 2 \cdot 4^2 + 5 \cdot 4 - 52 = 0.$$

COR. 2. Any rational integral function of x is divisible by $x - 1$, when the sum of the coefficients of the terms is zero.

For, putting $x = 1$ in the given function, it is evident that it is reduced to the sum of its coefficients, which sum must be zero if the function be divisible by $x - 1$.

EX. Each of the following functions is divisible by $x - 1$, viz.:—

$$\begin{aligned} &3x^4 + 7x^3 - x^2 + 12x - 21, \quad 5x^7 - 2x - 3, \\ &(a - b)x^2 + (b - c)x + (c - a), \quad (a + b)^2x^2 - 4abx - (a - b)^2. \end{aligned}$$

* The expression $f(x)$ must not be considered to mean the product of f and x , but as a *symbol* used for convenience.

COR. 3. Any rational integral function of x is divisible by $x + 1$, when the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers.

(The term independent of x is always to be considered as the coefficient of an *even* power).

Let $ax^n + ax^{n-1} + \&c. + rx^2 + sx + t$ be a rational integral function of x .

Put $x = -1$, then we have, if the function be divisible by $x + 1$ —

$$a(-1)^n + b(-1)^{n-1} + \&c. + r(-1)^2 + s(-1) + t = 0.$$

Suppose n to be even, then evidently $(-1)^n = (-1)(-1)(-1)\dots$ to an *even* number of factors $= +1$.

And so $(-1)^{n-1} = (-1)(-1)(-1)\dots$ to an *odd* number of factors $= -1$; and so on.

Hence we get $a - b + \&c. + r - s + t = 0$, and this must evidently require the condition that the sum of the positive quantities is equal to the sum of the negative, and, therefore, that the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers. And a similar result will follow if we suppose n to be *odd*.

EX. Each of the following functions is divisible by $x + 1$, viz:—

$$\begin{aligned} &x^3 + 5x^2 + 7x + 3, \quad 5x^5 - 4x^4 + 8x^3 - 2x - 1, \\ &a^2x^4 - (a+1)(a+2)x^3 + 2x + 3a + 4, \quad px^3 + (q+r)x^2 \\ &+ (q+r)x + p. \end{aligned}$$

EX. IX.

Resolve into elementary factors—

- $x^2 - 9a^2, 16y^4 - 25z^4, 24a^3 - 54b^3, 8x^3 - 27y^3.$
- $x^4 - xy^2, a^3 - b^3, xy^4 + x^2y, 2x^2y^2z - 8xy^2z^2.$
- $a^4 - 4b^4, x^4 + x^2y^2 + y^4, a^4 - 2a^2b^2 + b^4, a^3 + b^3 - c^3 + 2ab.$
- $a^2 + b^3 - c^2 - d^2 + 2ab - 2cd, a^2 - b^3 - c^3 + d^2 + 2bc + 2ad, a^2 - (b-c)^2.$
- $(x+7)^2 - (x+2)^2, (x+5)^2 - (x+2)^2, (2a+b)^2 - (a-c)^2.$
- $(x^3 - y^3)^2 + 4(x^4 + x^2y^2 + y^4)x^2y^2, (x^3 + y^3)^4 - 5(x^2 + y^2)^2x^2y^2 + 4x^4y^4.$

$$7. x^3 - 3x - 70, x^3 + 11x + 10, a^3 - 15ab + 56b^2, x^3 - 4x - 192.$$

$$8. a^2x^2 + abxy - 42b^2y^2, 3ax^2 - 24ax - 60a, 24ac - 5ac^2 + ac^3.$$

$$9. 6x^3 - 11x - 35, 8x^3 + 6x - 135, 18x^3 - 21x - 72, 20x^3 - 11x - 42.$$

$$10. 3x^2y + 10x^2y^2 + 3xy^3, 20x^3 + 12ax^2 + 25bx^2 + 15abx, m^2x^2 + (mq + mp)x + pq.$$

Write down the quotient of—

$$11. x^4 - 16 \text{ by } x - 2, 3x^5 + 96 \text{ by } x + 2, x^6 - 27 \text{ by } x^2 - 3.$$

$$12. (a + b)^3 - (c + d)^3 \text{ by } a + b + c + d, a^3 - b^3 + c^3 - d^3 + 2ac + 2bd \text{ by } a + b + c - d.$$

Find the remainder after the division of—

$$13. x^4 + px^3 + qx^2 + rx + s \text{ by } x - a, x^4 + a^4 \text{ by } x^2 - a^2.$$

$$14. x^3 - 5x^2 + 7x - 9 \text{ by } x + 3, x^4 - 3x + 7 \text{ by } x - 2.$$

Show that—

$$15. 5x^5 - 3x^3 + 7x^2 - 8x - 1 \text{ is divisible by } x - 1.$$

$$16. 2x^4 - 3x^3 + x^2 - 7x - 13 \text{ is divisible by } x + 1.$$

Show that :

$$17. a^3(b-c) + b^3(c-a) + c^3(a-b) = (a-b)(b-c)(a-c).$$

$$18. bc(b-c) + ac(c-a) + ab(a-b) = (a-b)(b-c)(a-c).$$

$$19. a^3(b-c) + b^3(c-a) + c^3(a-b) = (a-b)(b-c)(a-c) \\ (a + b + c).$$

$$20. a^4(b-c) + b^4(c-a) + c^4(a-b) = (a-b)(b-c)(a-c) \\ (a^2 + b^2 + c^2 + ab + ac + bc).$$

$$21. a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b) = (a+b+c)^3.$$

$$22. x^5 - y^5 = (x^3 - xy + y^2)(x^2 - xy + y^2)(x+y)(x-y).$$

$$23. x^5 - y^5 = (x^4 + y^4)(x^2 + y^2)(x+y)(x-y).$$

$$24. x^3 - 2xy - 3y^2 + 4yz - z^3 = (x - 3y + z)(x + y - z).$$

$$25. \ 3(x+y)^2 + 2(x+y)(z+u) - 5(z+u)^2 = (x+y-z-u)(3x+3y+5z+5u).$$

$$26. \ \frac{1}{6} \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \} \\ = \frac{1}{6} \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \} \cdot \frac{1}{3} \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \}.$$

$$27. \ \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \}^2 \\ = 2 \{ (x-y)^4 + (y-z)^4 + (z-x)^4 \}.$$

$$28. \ \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \} \cdot \frac{(x-y)^5 + (y-z)^5 + (z-x)^5}{5} \\ = \{ (x-y)^4 + (y-z)^4 + (z-x)^4 \} \cdot \frac{(x-y)^3 + (y-z)^3 + (z-x)^3}{3}.$$

$$29. \ x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2) = (x-y)(y-z)(z-x)(x+y)(y+z)(z+x).$$

$$30. \ x^{2n}-y^{2n} = (x-y)(x+y)(x^2-y^2)(x^4-y^4) \dots (x^{2n-1}+y^{2n-1}).$$

$$31. \ (a+b+c+d)^3 - (b+c-a-d)(c+a-b-d) \\ (a+b-c-d) = 4(a+b+c+d)(ab+bc+ac+ad+bd+cd) \\ - 8(abc+abd+acd+bcd).$$

CHAPTER III.

INVOLUTION AND EVOLUTION.

Involution.

31. Involution is the operation by which we obtain the powers of quantities. This can of course be done by multiplication, but the results obtained by the actual multiplication of simple forms enable us to develop without multiplication more complex forms. As the subject requires the aid of the Binomial Theorem, we shall here show how to develop a few only of the more simple expressions.

32. The power of a single term is obtained by raising the

coefficient of the term to the power in question, and *multiplying* the exponents of the letters of the term by the exponent of the power in question.

Thus, $(a^3)^2 = a^3 \times a^3 = a^{3+3} = a^6$;
 $(a^m)^n = a^m \times a^m \times a^m \dots \text{to } n \text{ factors} = a^{m+m+m+\dots \text{to } n \text{ terms}} = a^{mn}$.

And so, $(4a^2b^3c^4)^3 = 4^3 a^{2 \times 3} b^{3 \times 3} c^{4 \times 3} = 64 a^6 b^9 c^{12}$

33. Development of the *third, fourth, and fifth* powers of $a + b$.

We know (Art. 26, I.) that $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (a + b)^3 = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$;

also, $(a + b)^4 = (a^3 + 3a^2b + 3ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$;

and $(a + b)^5 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a + b) = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

The following law of the formation of the terms is evident:—

Law of Formation of Terms.

1. The first term contains a raised to the given power, and the power of a decreases by *unity* in each successive term, while the power of b (which first appears in the second term) increases by unity in each successive term, till it reaches the power of the given quantity.

2. The first coefficient is unity, and the coefficient of any term is found by multiplying the previous coefficient by the exponent of a in the previous term, and dividing the product by the number of terms hitherto developed.

Ex. 1. $(2x + 3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$.

Ex. 2. $(a + b + c)^3 = (a + b + c)^3 = (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3 = (a^3 + 3a^2b + 3ab^2 + b^3) + 3(a^2 + 2ab + b^2)c + 3(a + b)c^2 + c^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc$.

(In the following examples the above law may be *assumed* as generally true.)

Ex. X.

1. Find the values of $(a^2b^3)^2$, $(-3ab^3)^3$, $(2a^2bc)^4$, $(-x^2yz^3)^3$.

Expand—

2. $(a + 3b)^4$, $(2a + b)^4$, $(a - b)^5$, $(3a - 4b)^3$.
3. $(2m + 1)^4$, $(5x + 2)^3$, $(3a - 4c)^4$, $(-a - b)^3$.
4. $(x^2 + x + 1)^3$, $(3a - b + 4c - d)^3$, $(a + 2b - c)^3$
 $(3a + 3b + 3c)^3$.
5. $(1 + x - x^2)^3$, $(ax + by + cz)^3$, $\{(a + b)x - (c + d)y\}^3$.
6. $(1 + x)^7$, $(1 + x + x^2)^4$, $(a + bx + cx^2)^4$.
7. $(ax - by)^3$, $(3x + y)^3$, $(3x - y)^3$, $(x^2 + xy + y^2)^3$, $(x - y)^3$.
8. $(x^4 + x^2y^2 + y^4)^2$, $(x + y)^2$, $(x - y)^2$,
 $\{(a + b)^2 - 4ab\}^2$, $\{(a - b)^2 + 4ab\}^2$.

Simplify—

9. $(a + b + c)^3 - 3(a + b + c)^2c + 3(a + b + c)c^2 - c^3$.
10. $(a - b)^3 + 3(a - b)^2(b - c) + 3(a - b)(b - c)^2 + (b - c)^3$.
11. $(1 + x + 3x^2 + 3x^3)^3 + (1 - x + 3x^2 - 3x^3)^3$.
12. $\{(x + y)^3 - (x^2 + y^2)\}^3 - 27x^2y^3(x^3 + y^3)$.

Evolution.

34. Evolution is the operation by which we obtain the *roots* of quantities.

Since the *square* or *second power* of a^3 is $(a^3)^2$ or a^6 , we call (Art. 17) a^3 the *square root* or the *second root* of a^6 .

And so, since the *cube* or *third power* of a^3 is $(a^3)^3$ or a^9 , we call a^3 the *cube root* or the *third root* of a^9 .

So, generally, since the n th power of $a^m = (a^m)^n = a^{mn}$, we call a^m the n th root of a^{mn} .

Thus, we have $\sqrt{a^6} = a^3$, $\sqrt[3]{a^6} = a^2$, $\sqrt[n]{a^{mn}} = a^m$.

Hence, in the case of quantities consisting of a single letter with a given exponent, when the given exponent contains as a factor the *number* indicating the root, we must divide the *given exponent* by this number, the quotient being the exponent of the root.

(We shall see farther on that this rule holds when the given exponent is not so divisible, the root in this case being called a *surd*).

35. Since the product of an *even* number of *negative* factors must give a *positive* result, and the product of an *odd* number of *negative* factors a *negative* result, it follows that—

I. When the root is indicated by an *even* number—

1. The root of a *positive* quantity may be written either with a + or - sign.

$$\text{Thus, } \sqrt[2]{9a^2} = \pm 3a, \sqrt[4]{16b^4} = \pm 2b.$$

2. The root of a *negative* quantity is impossible.

$$\text{Thus, } \sqrt[2]{-a^2}, \sqrt[4]{-a^4b^4}, \&c., \text{ are impossible quantities.}$$

II. When the root is indicated by an *odd* number, the root has always the sign of the given quantity.

$$\text{Thus, } \sqrt[3]{-27b^3} = -3b, \sqrt[5]{32x^{10}y^5} = 2x^2y.$$

(It may be remarked that the theory of *impossible quantities* forms an important branch of Algebra, which the student cannot yet enter upon. According to that theory, all quantities have as many roots as the number indicating the root.)

Square Root.

36. We shall now develop the method of finding the square root of a given quantity.

Ex. 1. Find the square root of $a^2 + 2ab + b^2$.

We know that $a^2 + 2ab + b^2 = (a + b)^2$.

Hence, $a + b$ is the square root of $a^2 + 2ab + b^2$ or $a^2 + (2a + b)b$.

Now, it is evident that the first term a of the root is the square root of the first term a^2 of the given quantity; and if this term be subtracted, there remains $2ab + b^2$, from which to determine b the second term of the root. Now, b is contained in $2ab + b^2$ or $(2a + b)b$ exactly $(2a + b)$ times. Hence it follows that the second term of the root is found by dividing the remainder by twice the first term of the root, and, if we wish to arrange our work in a way similar to long division, it is evident that we first take for our divisor $2a + b$, that is, twice the first term of the root added to the second term, which multiplied by b the second term, and subtracted, leaves no remainder.

Thus the whole process may be arranged as follows :—

$$\begin{array}{r}
 a^2 + 2ab + b^2(a+b) \\
 a^2 \\
 \hline
 2a + b \quad \quad \quad 2ab + b^2 \\
 \quad \quad \quad \quad \quad 2ab + b^2
 \end{array}$$

Ex. 2. Find the square root of $a^3 + 2ab + b^3 + 2ac + 2bc + c^3$.

Now we know (Art. 26) that—

$$\begin{aligned}
 &a^3 + 2ab + b^3 + 2ac + 2bc + c^3 \text{ or } (a+b)^3 + \\
 &\quad 2(a+b)c + c^3 = (a+b+c)^3.
 \end{aligned}$$

And if we compare the form $(a+b)^3 + 2(a+b)c + c^3$ with the form $a^3 + 2ab + b^3$, it is evident that, having obtained as in the last example the first two terms $a+b$, we shall by continuing the process obtain the third term. Thus—

$$\begin{array}{r}
 a^3 + 2ab + b^3 + 2ac + 2bc + c^3(a+b+c) \\
 a^3 \\
 \hline
 2a + b \quad \quad \quad 2ab + b^2 \\
 \quad \quad \quad \quad \quad 2ab + b^2 \\
 \hline
 2a + 2b + c \quad \quad \quad 2ac + 2bc + c^3 \\
 \quad \quad \quad \quad \quad 2ac + 2bc + c^3
 \end{array}$$

We may deduce from the above examples the following general rule :—

RULE.

1. Arrange the terms of the given quantity according to the ascending or descending powers of some letter, and take the square root of the first term for the first term of the quotient.

2. Subtract the square of the quotient, and bring down the next two terms of the given quantity.

3. Double the quotient, and place the result as a *trial divisor*; then, dividing the first of the terms brought down by this trial divisor to obtain the second term of the root, add the quotient so obtained to the first term of the root, and also to the trial divisor, to obtain a *complete divisor*.

4. Multiply the complete divisor by the second term of the root, and subtract the product, as in long division, from the terms brought down.

5. If there be any remainder or more terms to bring down, double the whole quotient for a trial divisor, and divide the

remainder by the new trial divisor, to obtain the third term of the root; and so on.

Ex. 3. Find the square root of $1 - 4x + 10x^2 - 12x^3 + 9x^4$. The terms are here arranged according to the ascending powers of x . Then proceeding according to rule, we have—

$$\begin{array}{r}
 1 - 4x + 10x^2 - 12x^3 + 9x^4 \quad (1 - 2x + 3x^2) \\
 \underline{1} \\
 2 - 2x \qquad \qquad \qquad - 4x + 10x^2 \\
 \qquad \qquad \qquad \underline{- 4x + 4x^2} \\
 2 - 4x + 3x^2 \qquad \qquad \qquad 6x^2 - 12x^3 + 9x^4 \\
 \qquad \qquad \qquad \qquad \qquad \qquad \underline{6x^2 - 12x^3 + 9x^4}
 \end{array}$$

The student will observe that twice the quotient is most easily obtained by bringing down the previous complete divisor with its last term doubled.

Square Root of Numerical Quantities.

37. It is easy to apply the above method to numerical quantities.

Since $1^2 = 1$, $10^2 = 100$, $100^2 = 10,000$, $1,000^2 = 1,000,000$, &c it is evident that the square roots of numbers having less than three figures must contain one figure only;

That those having not less than three and less than five must contain two figures and two only;

Those having not less than five and less than seven must contain three figures and three only; and so on.

Hence it follows that, if a dot be placed over the units' figure, and over every alternate figure to the left, the number of dots will give the number of figures in the square root.

Thus, the square roots of the numbers $141\dot{3}7\dot{6}$ and $152\dot{2}75\dot{6}$ have three and four figures respectively.

In the number $141\dot{3}7\dot{6}$ we call 14, 13, 76 respectively the first, second, and third periods. So in the number $152\dot{2}75\dot{6}$, the first, second, third, and fourth periods are respectively 1, 52, 27, 56.

It is evident that the number of periods correspond to the number of figures in the square root, and it will be seen that the figures of each period are used in the operation for the corresponding figure of the root.

Ex. Find the square root of 565504.

Pointing off the number, we find the first period to be 56. Now, the greatest square in 56 is 49, and the square next greater is 64. Hence the number lies between the square of 700 and 800; and, following the algebraical method, 700 will be the first term of the root.

The operation will stand thus—

$$\begin{array}{r}
 565504 \text{ (} 700 + 50 + 2 = 752 \\
 490000 = a^2 \\
 2a + b = 1400 + 50 = 1450 \quad \overline{75504} \\
 72500 = 2ab + b^2 \\
 2a + 2b + c = 1400 + 100 + 2 = 1502 \quad \overline{3004} \\
 3004 = 2ac + 2bc + c^2
 \end{array}$$

Or, omitting the useless ciphers, and bringing down one period of figures at a time, the operation will stand thus—

$$\begin{array}{r}
 565504(752 \\
 49 \\
 145 \quad \overline{755} \\
 725 \\
 1502 \quad \overline{3004} \\
 3004
 \end{array}$$

Ex. 1. Find the square root of 6091024.

$$\begin{array}{r}
 6091024(2468 \\
 4 \\
 44 \quad \overline{209} \\
 176 \\
 486 \quad \overline{3310} \\
 2916 \\
 4928 \quad \overline{39424} \\
 39421
 \end{array}$$

Ex. 2. Find the square root of 83521.

$$\begin{array}{r}
 83521(289 \\
 4 \\
 48 \quad \overline{435} \\
 384 \\
 569 \quad \overline{5121} \\
 5121
 \end{array}$$

It will be observed that the second remainder, 51, is greater than the previous complete divisor, 48, and it might be supposed, therefore, that the second figure in the root should be 9 instead of 8.

Now, the square of $(a + 1)$ exceeds the square of a by $2a + 1$.

$$\text{Thus, } (a + 1)^2 - a^2 = (a^2 + 2a + 1) - a^2 = 2a + 1.$$

Hence it follows that, so long as any remainder is less than twice the corresponding number in the root + 1, we may be certain that we have taken the figure of the root sufficiently large.

Thus, since the remainder is less than $28 \times 2 + 1$ or 57, we may be certain that 8 is the correct figure and not 9.

Square Root of a Decimal.

38. It is evident that the *square* of any number containing one, two, three, &c., decimal figures, will contain *two, four, six, &c.*, decimal figures respectively ; and, hence, conversely, every decimal considered as a square must contain an even number of decimal figures, and its square root must contain half this even number of figures. It will then be necessary to add a cipher when the given number of decimal figures is odd.

Further, since decimals and integers follow the same system of notation, it is evident that if a dot be placed over the *units'* figure of the given number, the pointing off may be performed with regard to the integral part exactly as in integers, there being no necessity to point off the decimals, only taking care to bring them down in pairs, and putting a decimal point in the quotient when the first pair is brought down.

And again, if an integer be given which is not a perfect square, we may, by affixing to the right of it a decimal point and an even number of ciphers, gradually approximate to the square root as nearly as we please.

Ex. 3. Find the square root of 1.8225.

$$\begin{array}{r}
 1.8225(1.35 \\
 1 \\
 \hline
 23 \quad 82 \\
 \quad 69 \\
 \hline
 265 \quad 1325 \\
 \quad 1325 \\
 \hline
 \end{array}$$

Ex. 4. Find the square root of 247 to four places of decimals.

Instead of adding a decimal point and eight ciphers to the right of the given number, we will proceed in the ordinary way till we arrive at a remainder. Then putting a decimal point in the quotient, we shall add two ciphers to this and each successive remainder.

$$\begin{array}{r}
 247(15.7162 \\
 1 \\
 \hline
 25 \quad 147 \\
 \quad 125 \\
 \hline
 307 \quad 2200 \\
 \quad 2149 \\
 \hline
 3141 \quad 5100 \\
 \quad 3141 \\
 \hline
 31426 \quad 195900 \\
 \quad 188556 \\
 \hline
 314322 \quad 734400 \\
 \quad 628644 \\
 \quad 105756
 \end{array}$$

39. It will be shown hereafter that *when $n + 1$ figures of a square root have been obtained by the ordinary method, n figures more may be obtained by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is $2n + 1$.*

Ex. Find the square root of 29 to six places of decimals.

The square root required will evidently contain *seven* figures. We shall therefore find the first *four* figures by the ordinary method, and the other *three* by the above method. Thus—

	29	5·385164	
	25		
103	400		
	309		
1068	9100		
	8544		
10765	55600		
	53825		
10770	17750	then by division,	
	10770		
	69800		
	64620		
	51800		
	43080		
Ans. 5·385164.	8720		

Cube Root.

40. We will next develop the method of finding the cube root of a quantity.

Ex. 1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We know (Art. 33) that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Hence $a + b$ is the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We see then that, the quantity being arranged according to the powers of a , the first term a of the cube root is the cube root of the first term of the given quantity; and if this term a^3 be subtracted, there remains $3a^2b + 3ab^2 + b^3$.

We see again that if this remainder be divided by $3a^2$, its first term gives b the second term of the root, and, further, if it be divided by b , we get $3a^2 + 3ab + b^2$ as a quotient. If we wish therefore to arrange the whole process in a way

similar to ordinary division, it is evident that we must write as a divisor $3a^2 + 3ab + b^2$, in order that after multiplication by the quotient figure b we may obtain a quantity which when subtracted shall leave no remainder. The operation will then stand thus—

$$\begin{array}{r} a^3 + 3a^2b + 3ab^2 + b^3(a+b) \\ 3a^2 \overline{) 3a^3 + 3ab + b^3} \\ \underline{3a^2b + 3ab^2 + b^3} \end{array}$$

We call $3a^2$ the *trial divisor*, because by means of it we search for the second term of the cube root. Having obtained this second term, we then form the *complete divisor* $3a^2 + 3ab + b^2$.

Ex. 2. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$\begin{array}{r} 8x^3 - 36x^2y + 54xy^2 - 27y^3(2x - 3y) \\ 8x^3 \overline{) 8x^3 - 36x^2y + 54xy^2 - 27y^3} \\ \underline{- 36x^2y + 54xy^2 - 27y^3} \end{array}$$

EXPLANATION.—We find the cube root of $8x^3$ to be $2x$. This is then the first term of the quotient, and corresponds to a in the previous example. We now require $3a^2$ for a trial divisor. This, of course, $= 3(2x)^2 = 12x^2$. Subtracting the first term of the given quantity and dividing the first term of the remainder by this trial divisor, we obtain $-3y$ for the quotient. This forms the second term of the root, and corresponds to b in the last example. We now easily obtain $3ab$ and b^2 .

Thus, $3ab = 3(2x)(-3y) = -18xy$, and $b^2 = (-3y)^2 = 9y^2$.

Hence, the complete trial divisor, corresponding to $3a^2 + 3ab + b^2$ in the last example, $= 12x^2 - 18xy + 9y^2$.

Multiplying now by $-3y$ the quotient, we obtain $-36x^2y + 54xy^2 - 27y^3$, which subtracted leaves no remainder.

Hence, $2x - 3y$ is the cube root.

Ex. 3. Find the cube root of—

$$\begin{array}{r}
 x^6 + 6x^5y - 40x^3y^2 + 96xy^3 - 64y^6 \\
 \hline
 3x^4 + 6x^2y + 4x^2y^2 \\
 \hline
 3(x^2 + 2xy)^2(-4y^2) = 3x^4 + 12x^2y + 12x^2y^2 \\
 \quad - 12x^2y^2 - 24xy^3 \\
 \hline
 16y^4 \\
 \hline
 3x^4 + 12x^2y - 24xy^3 + 16y^4 \\
 \hline
 - 12x^2y^2 - 48x^2y^3 + 96xy^3 - 64y^6 \\
 \hline
 6x^2y^2 - 40x^2y^3 \\
 \hline
 6x^2y + 12x^4y^2 + 8x^2y^3 \\
 \hline
 x^6 + 6x^5y - 40x^3y^2 + 96xy^3 - 64y^6 \left\{ \begin{array}{l} x^2 + 2xy \\ - 4y^2 \end{array} \right.
 \end{array}$$

It will be evident that the first two terms are obtained exactly as in Exs. 1 and 2. To obtain the next term we treat the terms already found to a in Ex. 1. Then we obtain the trial divisor $3(x^2 + 2xy)^2$ corresponding to $3a^2$, and afterwards having divided the first term of the remainder by the first term of the trial divisor in order to get the term corresponding to b , we obtain in order $3(x^2 + 2xy)(-4y^2)$ corresponding to $3ab$, then $(-4y^2)^2$ corresponding to b^2 , and lastly, by addition, $3x^4 + 12x^2y - 24xy^3 + 16y^4$, being the complete divisor, corresponding to $3a^3 + 3ab + b^3$ of Ex. 1. We then conclude the operation as before.

Cube Root of Numerical Quantities.

41. We shall now apply the above method to numerical quantities. It may be shown, as in Art. 34, that, if we place a dot over the *units' figure*, and over every *third* figure to the left, the number of periods so formed will be the number of figures in the root.

Ex. 1. Find the cube root of 262144.

$$\begin{array}{r}
 26\dot{2}14\dot{4}(60 + 4 = 64 \\
 \underline{216000} \\
 46144 \\
 3a^2 = 10800 \\
 3ab = 720 \\
 b^2 = 16 \\
 \hline
 3a^2 + 3ab + b^2 = 11536 \qquad 46144
 \end{array}$$

EXPLANATION.—Pointing off the given number, we find the first period to be 262, and that the cube root consists of two figures. Now, the greatest perfect cube in 262 is 216, which is the cube of 6. Hence, the given number lies between the cubes of 60 and 70; and following the algebraical method, 60 will be the first term of the cube. This, we see, corresponds to a in the algebraical method.

We first then subtract the cube of a —viz., 216000, which leaves as a remainder 46144.

We now write down $3a^2$ or $3(60)^2 = 10800$, which is the trial divisor for determining b . Dividing then by this value of $3a^2$, we find $b = 4$, which is the second term of the cube root. We next obtain $3ab = 3(60)(4) = 720$, and $b^2 = 4^2 = 16$, and so by addition we get $3a^2 + 3ab + b^2 = 11536$, which is the complete divisor. Multiplying this then by the quotient figure, we subtract the product, and, there being no remainder, we find the cube root to be 64.

We may omit the useless ciphers in the above operation, if, remembering the local value of figures when numbers are expressed in ordinary notation, we take care to place the right-hand figure of the value of $3ab$ one place to the right of the corresponding figure of the value of $3a^2$; and also to place the right hand figure of b^2 one figure further still to the right.

The operation will then stand thus—

$$\begin{array}{r}
 26\dot{2}14\dot{4}(64 \\
 \underline{216} \\
 46144 \\
 3 \times 6^2 = 108 \\
 3 \times 6 \times 4 = 72 \\
 4^2 = 16 \\
 \hline
 11536 \qquad 46144
 \end{array}$$

Ex. 2. Find the cube root of 102503232.

$$\begin{array}{r}
 102503232 \overline{) 468} \\
 \underline{64} \\
 38503 \\
 3 \times 4^2 = 48 \\
 3 \times 4 \times 6 = 72 \\
 6^2 = 36 \\
 \hline
 5556 \\
 36 \\
 \hline
 33336 \\
 3 \times 46^2 = 6348 \\
 3 \times 46 \times 8 = 1104 \\
 8^2 = 64 \\
 \hline
 645904 \\
 5167232
 \end{array}$$

EXPLANATION.—The first two figures of the root are obtained as in Ex. 1. We then treat the number they form, viz., 46, as corresponding to a in the algebraical model, omitting useless ciphers. Obtaining then $3a^2$ or $3 \times 46^2 = 6348$, we find 8 to be the next figure of the root. Then writing under this, $3ab$ or $3 \times 46 \times 8 = 1104$, and afterwards b^2 or $8^2 = 64$, taking care as to the positions of the right-hand figures, and adding, we get 645904 as the complete divisor. Then as before.

REMARK.—It is unnecessary to be at the trouble to find the value of 3×46^2 by ordinary multiplication. For referring to the algebraical model, and writing here the successive terms of the complete divisor, and adding, we have—

$$\begin{array}{r}
 3a^2 \\
 3ab \\
 b^2 \\
 \hline
 \text{Sum} = 3a^2 + 3ab + b^2
 \end{array}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{If we now again write down } b^2 \\ \text{under this sum, and then add up} \\ \text{the last four lines, we get—} \end{array}$$

$$3a^2 + 6ab + 3b^2, \text{ or } 3(a^2 + 2ab + b^2) = 3(a + b)^2.$$

This is three times the square of the first two terms of the root.

It therefore follows that, if, as in the above example, after completing the operation for finding the first two figures of the cube root, we write under the complete divisor just obtained the value of the square of the second figure, and then add together the last four lines thus obtained, we get three

times the square of the quotient for a partial divisor by which to determine the next figure of the root.

The four lines to be added are in the above example bracketed. This method will be found to materially shorten the work, for it may be similarly applied to find the trial divisor when the cube root consists of any number of figures

Cube Root of a Decimal.

42. We know that the *cube* of any number containing one, two, three, &c., decimal figures will contain three, six, nine, &c., decimal figures respectively, and hence, conversely, every decimal considered as a cube must contain a number of decimal figures which is a multiple of *three*, and the number of decimal figures in the *cube root* must be *one-third* of the number contained in the given *cube*. It will then be necessary to add ciphers when the given number of decimal figures is not a multiple of 3.

And by continuing the reasoning of Art. 37, if a dot be placed over the units' figure and over every third figure to the left, it will be sufficient to bring down the decimal figures *three* at a time, putting a decimal point in the quotient when the first three are brought down.

And further, if an integer be given which is not a perfect cube, we may proceed in the ordinary way till we arrive at a remainder, and then, putting a decimal point in the quotient, by affixing *three* ciphers to this and each successive remainder, approximate to the cube root as nearly as we please.

Ex. 3. Find the cube root of 395·446904.

$$\begin{array}{r}
 3 \times 7^2 = 147 \\
 3 \times 7 \times 3 = 63 \\
 3^2 = 9 \\
 \hline
 15339 \\
 9 \\
 \hline
 3 \times 73^2 = 15987 \\
 3 \times 73 \times 4 = 876 \\
 4^2 = 16 \\
 \hline
 1607476
 \end{array}
 \left. \vphantom{\begin{array}{r} 147 \\ 63 \\ 9 \\ 15339 \\ 9 \\ 15987 \\ 876 \\ 16 \end{array}} \right\}
 \begin{array}{r}
 395\cdot446904(7\cdot34 \\
 343 \\
 \hline
 52446 \\
 \\
 46017 \\
 \hline
 6429904
 \end{array}$$

43. We shall show farther on that *when $n + 2$ figures of a cube root have been obtained by the ordinary method, n figures more may be obtained by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is $2n + 2$.*

We may apply this principle with advantage when we require the cube root of number to a given number of decimals.

Ex. Find the cube root of 87 to five places of decimals.

The required cube root will evidently contain 6 figures, and since 6 here corresponds to $2n + 2$ above, it is evident that $n = 2$. Hence, we shall find 4 (that is, $n + 2$) figures by the ordinary method, and then 2 more by division.

The operation will stand thus—

	87(4.43104
	64
	23000
$3 \times 4^2 = 48$	
$3 \times 4 \times 4 = 48$	
$4^2 = 16$	
}	21184
5296	1816000
16	
$3 \times 44^2 = 5808$	
$3 \times 44 \times 3 = 396$	
$3^2 = 9$	
}	1754307
584769	61693000
9	
$3 \times 443^2 = 588747$	
$3 \times 443 \times 1 = 1329$	
$1^2 = 1$	
}	58887991
58887991	280500900
1	235605132
58901283	44895768

Ans. 4.43104.

Ex. XI.

Find the square roots of—

1. $4x^2y^4z^6$, $16a^4y^4$, $a^4 + 2a^2x^2 + a^4$.
2. $4x^5 - 12x^3y + 29x^4y^2 - 30x^2y^3 + 25x^2y^4$.

3. $25a^4 - 30a^3b + 19a^2b^2 - 6ab^3 + b^4$.
 4. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.
 5. $a^3 + 2abx + (2ac + b^2)x^2 + 2(ad + bc)x^3 + (2bd + c^2)x^4 + 2cdx^5 + d^2x^6$.
 6. $a^4x^{2n} - 6a^3x^{2n-1} + 17a^2x^{2n-2} - 24ax^{2n-3} + 16x^{2n-4}$.
 7. $x^2 + 2 + x^{-2}, a^2x^{-2} - 2 + a^{-2}x^2$.
 8. $9x^{2m} - 3a^2x^m + 25a^3 - 30ax^m + \frac{a^4}{4} + 5a^2$.

Find the square roots of—

9. 1296, 6241, 42849, 83521.

10. 10650.24, .000576, $\frac{1}{243}$.

11. $\sqrt{17}, \sqrt{1.5}, \frac{2}{\sqrt{5}+1}, \frac{\sqrt{3}+1}{\sqrt{3}-1}$.

Give the values correct to four places of decimals of—

12. $\frac{\frac{4}{9} \text{ of } .31416}{\sqrt{93}}, \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} + \frac{\sqrt{10} - 2}{\sqrt{10} + 2}, 3.1416 \text{ of}$

$\sqrt{\frac{4}{32.16}}$.

Find the cube roots of—

13. $8a^3b^6y^3, 125x^{12}y^3, a^3 + 6a^2b + 12ab^2 + 8b^3$.
 14. $x^{12} + 9x^{10} + 6x^8 - 99x^6 - 42x^4 + 441x^2 - 343$.
 15. $x^3 + 3x^2y + 3xy^2 + y^3 - 6axy - 3cx^2 - 3cy^2 + 3c^2x + 3c^2y - c^3$.
 16. $x^3 + x^{-3} + 3(x + x^{-1}), x^3y^{-3} + 3x^2y^{-3} + 3xy^{-1} + 1$.

Find the cube roots of—

17. 5849513501832, 1371.330631.

18. 20.346417; .037, $\frac{1}{1372}$.

Give the value of the following correct to four places of decimals :—

19. $\frac{\sqrt[3]{5.12} + \sqrt[3]{.03375}}{\sqrt[3]{80} - \sqrt[3]{.01}}, \frac{1}{\sqrt[3]{4} + \sqrt[3]{2} + 1}$.

20. $\frac{\sqrt[3]{.625} + \sqrt[3]{.04}}{\sqrt[3]{.05} + \sqrt[3]{.04}} \text{ of } \frac{\sqrt{5} + 2}{7}$.

21. $(\sqrt{7} + 2)(\sqrt{7} - 1), (5 + \sqrt{3})(4 + \sqrt{12})$.

$$22. \sqrt{11 + 6\sqrt{2}}, \sqrt[3]{6 + 15\sqrt{3}}.$$

$$23. \sqrt[3]{\frac{3}{4} - \frac{6 + 2\sqrt{5}}{16}} : \frac{\sqrt{5} + 1}{4}.$$

$$24. a^3(b - c) - b^3(a - c) + c^3(a - b), \text{ where } a = \sqrt{12}, \\ b = -\sqrt{3}, \text{ and } c = -\sqrt[3]{027}.$$

CHAPTER IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

Greatest Common Measure.

44. In Arithmetic (page 24) we defined the G.C.M. of two or more numbers as their highest common factor. In Algebra the same definition will suffice, provided we understand by the term *highest common factor*, the factor of highest dimensions (Art. 18). This, it need hardly be remarked, does not necessarily correspond to the factor of highest numerical value.

45. To find the G.C.M. of two quantities.

RULE.—Let A and B be the quantities, of which A is not of lower dimensions than B. Divide A by B, until a remainder is obtained of lower dimensions than B. Take this remainder as a new divisor, and the preceding divisor A as a new dividend, and divide till a remainder is again obtained of lower dimensions than the divisor; and so on. The last divisor is the G.C.M.

Before giving the general theory of the G.C.M. we shall work out a few examples.

Ex. 1. Find the G.C.M. of $x^3 - 6x - 27$ and $2x^3 - 11x - 63$.

According to the above rule, the operation is as follows:—

$$\begin{array}{r} x^3 - 6x - 27 \quad 2x^3 - 11x - 63(2 \\ \underline{2x^3 - 12x - 54} \\ x - 9) \quad x^3 - 6x - 27(x + 3 \\ \underline{x^3 - 9x} \\ 3x - 27 \\ \underline{3x - 27} \\ 0 \end{array}$$

$\therefore$ The G.C.M. is $x - 9$.

Ex. 2. Find the G.C.M. of $10x^3 + 31x^2 - 63x$ and $14x^3 + 51x^2 - 54x$.

We may tell by inspection that x is a common factor, which we therefore strike out of both, only *taking care to reserve* it. The quantities then become—

$$10x^2 + 31x - 63, \text{ and } 14x^2 + 51x - 54.$$

We may now proceed according to rule, taking the former as divisor. We see, however, that the coefficient of the first term of the dividend is not exactly divisible by the coefficient of the first term of the divisor. Multiply therefore (to avoid fractions) the dividend by such a number as will make it so divisible, viz., by 5. This will not affect the G.C.M., as 5 is not a *factor* of the first expression, viz., $10x^2 + 31x - 63$.

It may as well be here mentioned that the G.C.M. of two quantities cannot be affected by the multiplication or division of *one* of the quantities by any quantity which is not a measure of the other. We shall, for a similar reason, reject certain factors or introduce them into any of the remainders or dividends during the operation. (See Art. 47).

$$\begin{array}{r}
 14x^2 + 51x - 54 \\
 5 \\
 \hline
 10x^2 + 31x - 63 \quad 70x^2 + 255x - 270 \quad 7 \\
 \hline
 70x^2 + 217x - 441 \\
 \hline
 38x + 171
 \end{array}$$

Rejecting the factor 19 of this remainder, we have—

$$\begin{array}{r}
 2x + 9 \quad 10x^2 + 31x - 63 \quad 5x - 7 \\
 \hline
 10x^2 + 45x \\
 \hline
 - 14x - 63 \\
 \hline
 - 14x - 63 \\
 \hline
 \hline
 \end{array}$$

Hence, $2x + 9$ is the last divisor, and multiplying this by x , the common measure struck out at the commencement, we find the G.C.M. to be $x(2x + 9)$ or $2x^2 + 9x$.

Ex. 3. Find the G.C.M. of $x^5 - 7x^4 - 3x^3 - 5x^2 + 42x - 21$, and $x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21$.
 $x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21$ $x^5 - 7x^4 - 3x^3 - 5x^2 + 42x - 21$
 $4x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21$ $4x^5 - 28x^4 + 91x^3 - 13x - 21$

$$\begin{array}{r} 4x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21 \\ 4x^5 - 44x^4 + 100x^3 + 76x^2 - 196x - 84 \\ \hline 16x^4 - 124x^3 + 15x^2 + 183x + 63 \end{array}$$

Multiplying the preceding divisor by 16, and taking the result for a dividend, we have—

$$\begin{array}{r} 16x^4 - 124x^3 + 15x^2 + 183x + 63 \\ 16x^4 - 176x^3 + 400x^2 + 304x^2 - 784x - 336(x) \\ \hline 16x^5 - 124x^4 + 15x^3 + 183x^2 + 63x \end{array}$$

$$\begin{array}{r} - 52x^4 + 385x^3 + 121x^2 - 847x - 336 \\ 4 \end{array}$$

(Multiplying this remainder by 4)

$$\begin{array}{r} - 208x^4 + 1540x^3 + 484x^2 - 3388x - 1344(-13) \\ - 208x^4 + 1612x^3 - 195x^2 - 2379x - 819 \\ \hline - 72x^3 + 679x^2 - 1009x - 525- \end{array}$$

Multiplying the preceding divisor by 9, and taking the result for a dividend, we have—

$$\begin{array}{r} - 72x^3 + 679x^2 - 1009x - 525 \\ 144x^4 - 1358x^3 + 2018x^2 + 1050x \end{array}$$

$$\begin{array}{r} 242x^3 - 1883x^2 + 597x + 567 \\ 36 \end{array}$$

(Multiplying this remainder by 36)

$$\begin{array}{r} 8712x^3 - 67788x^2 + 21492x + 20412(121) \\ 8712x^3 - 82159x^2 + 122089x + 63526 \\ \hline 14371x^2 - 100597x - 43113 \end{array}$$

Dividing this remainder by 14371, and taking the quotient for a new divisor, we have—

$$\begin{array}{r}
 x^2 - 7x - 3) - 72x^2 + 679x - 1009x - 525(-72x + 175 \\
 \quad - 72x^2 + 504x^2 + 216x \\
 \hline
 \quad \quad 175x^2 - 1225x - 525 \\
 \quad \quad 175x^2 - 1225x - 525 \\
 \hline
 \end{array}$$

$\therefore x^2 - 7x - 3$ is the G.C.M.

It will be seen that we have introduced and rejected factors during the operation in order to avoid fractional coefficients. This, as will be seen from the general theory, will not affect the result, provided that no factor thus introduced or rejected is a measure of the corresponding divisor or dividend, as the case may be.

Theory of the Greatest Common Measure.

46. Let A and B be the two algebraical quantities, and the operation as indicated by the rule (Art. 45) be performed. Thus, let A be divided by B , with quotient p and remainder C . Then let B be divided by C , with quotient q , and remainder D . Lastly, let C be divided by D , with quotient r , and remainder zero.

$$\begin{array}{r}
 B)A(p \\
 \quad pB \\
 \hline
 \quad C)B(q \\
 \quad \quad qC \\
 \hline
 \quad \quad D)C(r \\
 \quad \quad \quad rD \\
 \quad \quad \quad \quad 0
 \end{array}$$

Then we are required to show that D is the G.C.M. of A and B .

(1.) D is a common measure of A and B .

Now, we have $C = rD$, $B = qC + D$, $A = pB + C$. Hence, D is a measure of C , and therefore of qC . It is therefore a measure of $qC + D$ or B . Hence, also, D is a measure of pB , and since it is also a measure of C , it must be a measure of $pB + C$ or A . But we have shown it to be a measure of B . Hence, D is a common measure of A and B .

(2.) D is the G.C.M. of A and B .

For every measure of A and B will divide $A - pB$ or C ; and hence every measure of A and B will divide $B - qC$ or D . Now, D cannot be divided by any quantity higher than D , and, therefore, there cannot exist a measure of A and B higher than D . Hence, D is the G.C.M. of A and B .

47. A factor which does not contain any factor common to both A and B may be rejected at any stage of the process.

Let the operation stand thus:—

$$\begin{array}{r} B = mB' \text{ suppose,} \\ B')A(p \\ \underline{pB} \\ C = nC' \text{ suppose,} \end{array} \quad \begin{array}{r} C')B'(q \\ \underline{qC'} \\ D)C'(r \\ \underline{rD} \\ 0 \end{array}$$

where neither m nor n contains any factor common to A and B .

It will be an exercise for the student to show that D is the G.C.M. of A and B .

48. A factor, which has no factor that the divisor has, may be introduced into the dividend at any stage of the process.

The operation may stand thus—

$$\begin{array}{r} B)mA(p, \text{ where } m \text{ has no factor that } B \text{ has;} \\ pB \\ C)nB(q, \text{ where } n \text{ has no factor that } C \text{ has;} \\ \underline{qC} \\ D)C'(r \\ \underline{rD} \\ 0 \end{array}$$

As in Arts. 46, 47, it may be easily shown that D is the G.C.M.

Both the above principles are made use of in working out Ex. 3 Art. 45.

49. When a common factor can be found by inspection, it is advisable to strike it out of the given expressions. Then, having found by the ordinary process the G.C.M. of the resulting quantities, we must multiply the G.C.M. so found by the rejected factor.

Thus, $4x$ is common to the quantities $4x^3 - 20x^2 + 24x$, and $4x^3 + 16x^2 - 84x$.

Rejecting it, we get $x^3 - 5x + 6$, and $x^3 + 4x - 21$, whose G.C.M. is easily found to be $x - 2$.

Multiplying by $4x$, we find the required G.C.M. to be $4x^3 - 8x$.

50. By a little ingenuity on the part of the student in breaking up the given expressions into factors, the ordinary and often tedious process of finding the G.C.M. may be avoided. The limits of our space will allow us only one example.

Ex. Find the G.C.M. of $3x^3 + 4x^2 - 10x + 3$, and $15x^3 + 47x^2 + 13x - 12$.

The first expression contains $x - 1$ as a factor (Art. 30), for the sum of its coefficients is *zero*. The other factor may be obtained thus—

$$\begin{aligned} 3x^3 + 4x^2 - 10x + 3 &= 3x^3 - 3x^2 + 7x^2 - 7x - 3x + 3 \\ &= 3x^2(x - 1) + 7x(x - 1) - 3(x - 1) \\ &= (3x^2 + 7x - 3)(x - 1). \end{aligned}$$

Now, $3x^2 + 7x - 3$ is not further resolvable, and $x - 1$ is evidently (Art. 30) not a factor of $15x^3 + 47x^2 + 13x - 12$. It is, therefore, very probable that $3x^2 + 7x - 3$ is the G.C.M. required.

We may test it thus—

$$\begin{aligned} 15x^3 + 47x^2 + 13x - 12 &= 15x^3 + 35x^2 - 15x + 12x^2 + 28x - 12 \\ &= 5x(3x^2 + 7x - 3) + 4(3x^2 + 7x - 3) \\ &= (5x + 4)(3x^2 + 7x - 3). \end{aligned}$$

Hence, $3x^2 + 7x - 3$ is the G.C.M. required.

G.C.M. of Three or More Quantities.

51. The G.C.M. of three or more quantities may be found thus—

RULE.—Find the G.C.M. of any two of the quantities, then the G.C.M. of the G.C.M. so found and a third quantity, and so on. The last found G.C.M. will be the G.C.M. required.

Ex. XII.

Find the G.C.M. of the following—

1. $x^3 - 5x + 6$ and $x^3 + 3x - 18$.
2. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 5x^2 + 7x + 3$.
3. $2x^3 + 10x^2 - 18x - 90$ and $3x^3 + 16x^2 - 26x - 141$.
4. $x^2 + (a + b)x + ab$ and $x^2 + (a + c)x + ac$.
5. $a^3 - b^3$ and $a^3 + a^2b + ab^2$.

6. $x^3 - 4x + 3$ and $x^3 + 4x^2 - 5$.
7. $4x^3 - 32x^2 + 85x - 75$ and $3x^3 - 15x^2 + 15x + 9$.
8. $9x^3 - 3xy + 2y - 4$ and $6x^4 - 4x^3 - 9xy^2 + 6y^2$.
9. $48x^4 + 8x^3 + 31x^2 + 15x$ and $24x^4 + 22x^2 + 17x^3 + 5x$.
10. $15a^3 + a^2b - 3ab^2 + 2b^3$ and $54a^2b^2 - 24b^4$.
11. $3x^3 - (3c + d + 1)x^2 - (2a + b - 3c - d + 2)x + 2a + b$ and $2x^3 - (a + b + 2)x + a + b$.
12. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.
13. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ and $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$.
14. $e^2x^3 + e^2 + x^3 + 1$ and $e^{2x}x^4 - e^{2x} + x^4 - 1$.
15. $ax^4 + (b + c)x^3 - ax - b - c$ and $ex^3 - (f - g)x^2 + (f - e)x - g$.
16. $4x^4 + 2x^3 + 4x^2 + 39x - 9$, $8x^4 + 20x^3 + 51x^2 + 9$, and $2x^4 + x^3 + 3x^2 + 18x$.
17. $ax^3 - (c + 1)x^2 + (c + 1)x - a$, $bx^4 - (b + d)x^3 + (c + d)x^2 - (c + e)x + e$, and $(c + 1)x^5 + (d + 2)x^4 - (d + 1)x^3 - (c + 2)x^2$.
18. $a^3 - b^3 + c^3 + 3abc$ and $a^3 - b^3 + c^3 + 2ac$.

Least Common Multiple.

52. When two or more algebraical expressions are arranged according to the powers of some letter, the expression of lowest dimensions which is divisible by each of the given expressions is called the L.C.M.

53. The L.C.M. of monomials and of expressions whose factors are apparent may be found by inspection.

Ex. 1. Find the L.C.M. of ab , ac , ad , bc , bd , cd .

If we form an expression, whose elementary factors contain each of the elementary factors of the given quantities, we shall evidently have a *common multiple*; and if no elementary factor of this expression is of a higher power than the highest power of the same factor in the given quantities, we shall get the L.C.M.

Hence, the required L.C.M. = $abcd$.

Ex. 2. Find the L.C.M. of—

$$(a - b)(b - c), (a - b)(c - a), (b - c)(c - a).$$

Ans. $(a - b)(b - c)(c - a)$.

Ex. 3. Find the L.C.M. of $a(x + 1)$, $b(x^2 - 1)$, $c(x^2 + 2x - 3)$, $d(x^2 + 4x - 3)$. We may write the given expressions thus—

$$a(x + 1), b(x + 1)(x - 1),$$

$$c(x - 1)(x + 3), d(x + 1)(x + 3).$$

Hence, the required L.C.M. = $abcd(x - 1)(x + 1)(x + 3)$.

Ex. 4. Find the L.C.M. of $a^2 - ax + x^2$, $a^2 + ax + x^2$, $a^3 + x^3$, $a^3 - x^3$.

$$\text{Now (Art. 29) } a^3 + x^3 = (a + x)(a^2 - ax + x^2),$$

$$\text{and } a^3 - x^3 = (a - x)(a^2 + ax + x^2).$$

Hence the required L.C.M.—

$$= (a + x)(a - x)(a^2 + ax + x^2)(a^2 - ax + x^2) = a^6 - x^6.$$

54. *The L.C.M. of two quantities is found by dividing their product by their G.C.M.*

Let a and b be the two quantities, and d the G.C.M. ;

And suppose $a = pd$ and $b = qd$.

It is evident that p and q contain no common factor. Hence pq is the L.C.M. of p and q ; and, therefore, no expression of lower dimensions than pqd can possibly be divisible by pd and qd .

Hence pqd is the L.C.M. of pd and qd , or of a and b .

Now $pqd = pd \times qd \div d = a \times b \div d$, and hence the rule.

55. To find the L.C.M. of *three* or more quantities.

RULE.—Find the L.C.M. of two of the quantities, then the L.C.M. of the expression thus obtained and a third quantity, and so on. The last expression so found is the L.C.M. required.

We shall prove this rule in the case of *three* quantities.

Let a , b , c be the quantities, and m be the L.C.M. of a and b .

Then the L.C.M. of m and c is the L.C.M. required.

For every common multiple of m and c is a common multiple of a , b , c . And every common multiple of a and b

must contain the m , their least common multiple. Hence, every common multiple of a, b, c must be a common multiple of m and c , and the converse is also true. Hence, the L.C.M. of m and c is the L.C.M. of a, b, c .

Ex. XIII.

Find the L.C.M. of—

1. $axy^2, 3a^2xy^2, 4a^3y^2, 6x^2y^2$.
2. $5a^2b^2, 6a^2c^2, 4b^2c^2$.
3. $(a - b)(b - c), (b - a)(a - c), (c - a)(c - b)$.
4. $ax(x + a), a^2(x - a), x^2 - a^2$.
5. $x^2 + 3x + 2, x^2 + 4x + 3, x^2 + 5x + 6$.
6. $x^2 - x - 30, x^2 - 11x + 30, x^2 - 25$.
7. $6x^2 + 37x + 56, 8x^2 + 38x + 35, 12x^2 + 47x + 40$.
8. $5(x^2 - x + 1), 6(x^2 + 1), 7(x^2 + 1)$.
9. $x^4 + a^2x^2 + a^4, a^2x^2 + a^3x + a^4, ax^2 - a^2x + a^3$.
10. $x^2 + (a + b)x + ab, x^2 + (a + c)x + ac, x^2 + (b + c)x + bc$.
11. $1 - x, 1 + x, 1 + x^2, 1 + x^4, 1 + x^8$.
12. $x^3 + 6x^2 + 11x + 6, x^3 - 6x^2 - 25x + 150$.
13. $a^3 - 3ab(a - b) - b^3, a^3 - b^3, a^3 + a^2b + ab^2$.
14. $x^4 - 1, 6x^3 + 5x^4 + 8x^2 + 4x^3 + 2x - 1$.
15. $a^4 - 2a^2b^2 + b^4, a^4 + 4a^2b + 6a^2b^2 + 4ab^3 + b^4, a^4 - 4a^2b + 6a^2b^2 - 4ab^3 + b^4$.
16. $3x^3 - 4x + 1, 2x^3 - 7x + 5, 4x^4 + 6x^2 + 10x$.
17. $3x^2 + 6x - 24, x^3 - 12x + 16, 5x^4 - 22x - 36$.
18. $a^3 - ab^2, b^3 - a^2b, ab^2 - b^3, a^2b - a^3$.
19. $3x^4 - 48, 5x^3 - 20, 3x^2 - 16x + 20$.
20. $x^5 - y^5, x^4 - 2x^2y^2 + y^4, x^3 + x^2y + xy^2 + y^3$.
21. $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$ and $x^5 - ax^4 + a^2x^3 - a^3x^2 + a^4x - a^5$.
22. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$ and $a^2 - b^2 - c^2 + d^2 + 2ad - 2bc$.

$$23. a^3 + b^3 + c^3 - 3abc \text{ and } (a + b)^3 + 2(a + b)c + c^3.$$

$$24. (a + b)^3 - (c + d)^3, (a + c)^3 - (b - d)^3, (a + d)^3 - (b - c)^3, (c + d)^3 - (a - b)^3, (b + d)^3 - (a - c)^3, \text{ and } (b + c)^3 - (a - d)^3.$$

CHAPTER V.

Fractions.

56. It is unnecessary to repeat here the propositions relating to fractions which were proved in Arithmetic, Chap. II. of this work. The student will see that, by substituting general symbols for the particular figures there used, the reasoning will equally hold. We shall work out a few examples to show the method of dealing with them in algebra.

Ex. 1. Simplify the fraction $\frac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15}$.

By inspection (Art. 30) we see that $x - 3$ is a factor of numerator and denominator. We have then—

$$\begin{aligned} \frac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15} &= \frac{x^2(x - 3) + x(x - 3) + 4(x - 3)}{(x - 3)(x + 5)} \\ &= \frac{(x^2 + x + 4)(x - 3)}{(x - 3)(x + 5)} = \frac{x^2 + x + 4}{x + 5}. \quad \text{Ans.} \end{aligned}$$

Ex. 2. Find the value of $\frac{1}{a + b} + \frac{1}{a - b} - \frac{2a}{a^2 + b^2}$.

$$\begin{aligned} \frac{1}{a + b} + \frac{1}{a - b} - \frac{2a}{a^2 + b^2} &= \frac{(a - b) + (a + b)}{(a + b)(a - b)} - \frac{2a}{a^2 + b^2} \\ &= \frac{2a}{a^2 - b^2} - \frac{2a}{a^2 + b^2} = 2a \left(\frac{1}{a^2 - b^2} - \frac{1}{a^2 + b^2} \right) \\ &= 2a \cdot \frac{(a^2 + b^2) - (a^2 - b^2)}{(a^2 - b^2)(a^2 + b^2)} = 2a \cdot \frac{2b^2}{a^4 - b^4} = \frac{4ab^2}{a^4 - b^4}. \end{aligned}$$

Ex. 3. Find the value of $\frac{bc}{(a-b)(a-c)} + \frac{ac}{(b-a)(b-c)} + \frac{ab}{(c-a)(c-b)}$.

The second denominator has a factor, $(b-a)$, which differs from a factor, $(a-b)$, of the first denominator in sign only. We shall therefore change the sign of the second fraction, and also of its first factor. This will not alter its value.

And, similarly, we find that by changing the signs of each of the factors of the third denominator we shall have them in a form corresponding to factors of the first and second denominators. The sign of the third fraction will not be changed, as the sign of the denominator will, on the whole, be unchanged.

The given expression then will stand thus—

$$\begin{aligned} &= \frac{bc}{(a-b)(a-c)} - \frac{ac}{(a-b)(b-c)} + \frac{ab}{(a-c)(b-c)} \\ &= \frac{bc(b-c) - ac(a-c) + ab(a-b)}{(a-b)(a-c)(b-c)} \\ &= \frac{bc(b-c) - a^2c + ac^2 + a^2b - ab^2}{(a-b)(a-c)(b-c)}, \text{ or, re-arranging,} \\ &= \frac{a^2(b-c) - a(b^2 - c^2) + bc(b-c)}{(a-b)(a-c)(b-c)}, \text{ then, dividing nume-} \\ &\text{rator and denominator by } b-c, \\ &= \frac{a^2 - a(b+c) + bc}{(a-b)(a-c)} = \frac{(a-b)(a-c)}{(a-b)(a-c)} = 1. \end{aligned}$$

Ex. 4. Simplify—

$$\left\{ a^2 - \frac{4a^2x - 3ax^2 + x^3}{a+x} \right\} \times \left\{ a^2 + \frac{4a^2x + 3ax^2 + x^3}{a-x} \right\}.$$

The given expression—

$$\begin{aligned} &= \frac{a^2(a+x) - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^2(a-x) + 4a^2x + 3ax^2 + x^3}{a-x} \\ &= \frac{a^3 + a^2x - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^3 - a^2x + 4a^2x + 3ax^2 + x^3}{a-x} \end{aligned}$$

$$\begin{aligned}
&= \frac{a^3 - 3a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a-x} \\
&= \frac{(a-x)^3}{a+x} \times \frac{(a+x)^3}{a-x} = \frac{(a-x)^3}{1} \times \frac{(a+x)^3}{1} = (a^3 - x^3)^2.
\end{aligned}$$

Ex. 5. Divide—

$$(a+b) \left\{ \frac{1}{x-a} + \frac{1}{x+b} \right\} \text{ by } (a-b) \left\{ \frac{a}{(x-a)^2} + \frac{b}{(x+b)^2} \right\}.$$

Now—

$$\frac{(a+b) \left\{ \frac{1}{x-a} - \frac{1}{x+b} \right\}}{(a-b) \left\{ \frac{a}{(x-a)^2} + \frac{b}{(x+b)^2} \right\}} = \frac{(a+b) \cdot \frac{(x+b) - (x-a)}{(x-a)(x+b)}}{(a-b) \cdot \frac{a(x+b)^2 + b(x-a)^2}{(x-a)^2(x+b)^2}}$$

or, reducing—

$$\begin{aligned}
&= \frac{(a+b) \cdot \frac{a+b}{(x-a)(x+b)}}{(a-b) \cdot \frac{(a+b)x^2 + ab(a+b)}{(x-a)^2(x+b)^2}} = \frac{(a+b) \cdot \frac{1}{1}}{(a-b) \cdot \frac{x^2 + ab}{(x-a)(x+b)}} \\
&= \frac{(a+b)(x-a)(x+b)}{(a-b)(x^2 + ab)}.
\end{aligned}$$

Ex. XIV.

Simplify the following expressions:—

1. $\frac{x^2 - 5x + 4}{x^3 + 2x - 24} \cdot \frac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}$
2. $\frac{6x^2 + 29x + 35}{14x^2 + 39x + 10} \cdot \frac{2x^3 + 7x - 9}{5x^3 - 3x^2 - 4x + 2}$
3. $\frac{a^4 - 2a^2b^2 + b^4}{a^3 - 4a^2b + 4ab^2 - b^3} \cdot \frac{24a^3 - 28a^2b + 6ab^2 - 7b^3}{6a^2 + 11ab - 21b^2}$
4. $\frac{x^2(y^2 - z^2) - xy(2y^2 + yz - z^2) + y^3(y + z)}{x^3(y + z)^2 - xy(2y^2 + 3yz + z^2) + y^3(y + z)} \cdot \frac{x^4 + x^2y^2 + y^4}{x^6 - y^6}$
5. $\frac{1}{a+b} + \frac{1}{a-b} \cdot \frac{1}{a-b} - \frac{1}{a+b}$

6. $\frac{a}{a+b} + \frac{b}{a-b} - \frac{ab}{ab-b^2} + \frac{ab}{a^2+ab}$.
7. $\frac{1}{4(x-1)} - \frac{x-1}{4(x^2+1)} - \frac{x-1}{2(x^2+1)^2}$.
8. $\frac{5}{x+1} - \frac{2}{(x+2)^2} - \frac{18}{5(x+2)} - \frac{7x+1}{5(x^2+1)}$.
9. $\frac{11}{x+1} - \frac{11}{x+3} - \frac{14}{(x+1)^2}$.
10. $\frac{8}{(x-1)^4} - \frac{4}{(x-1)^3} - \frac{2}{(x-1)^2} + \frac{16x+14}{3(x^2-1)} - \frac{16x-8}{3(x^2-x+1)}$.
11. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$.
12. $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)}$.
13. $\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)}$.
14. $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$.
15. $\frac{1}{\left(\frac{a}{b}-1\right)\left(\frac{a}{c}-1\right)} + \frac{1}{\left(\frac{b}{a}-1\right)\left(\frac{b}{c}-1\right)} + \frac{1}{\left(\frac{c}{a}-1\right)\left(\frac{c}{b}-1\right)}$.
16. $\frac{(a+b)^2 + (b-c)^2 + (a+c)^2}{(a+b)(b-c)(a+c)} - \frac{2}{a+c} - \frac{2}{b-c} + \frac{2}{a+b}$.
17. $\left\{ \frac{1}{a+x} + \frac{1}{a-x} + \frac{2a}{a^2+x^2} \right\} \cdot \left\{ \frac{1}{a+x} - \frac{1}{a-x} - \frac{2x}{a^2+x^2} \right\}$.
18. $\left\{ \frac{a^2+b^2}{a^2-b^2} + \frac{a-b}{a+b} \right\} \div \left\{ 2 + \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b} \right\}$.
19. $\left\{ 1 - \frac{2x^2}{a^2} + \frac{2x^4}{a^2(a^2+x^2)} \right\} \times \left\{ \frac{2a^4}{x^2(a^2-x^2)} - \frac{2a^2}{x^2} - 1 \right\}$.
20. $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} - \frac{a^2\left(\frac{b}{c}+1\right) + b^2\left(\frac{c}{a}+1\right) + c^2\left(\frac{a}{b}+1\right)}{ab+bc+ca}$.

$$21. \left\{ \left(\frac{x+y}{x-y} \right)^2 + 1 \right\} \cdot \left\{ \left(\frac{x+z}{x-z} \right)^2 + 1 \right\} \cdot \left\{ \left(\frac{y+z}{y-z} \right)^2 + 1 \right\} \times \\ \frac{x^2(y-z) + y^2(z-x) + z^2(x-y)}{x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4 + 2x^2y^2z^2}.$$

$$22. \left\{ \frac{1-x^2}{1-x^3} + \frac{1-x}{1-x+x^3} \right\} \div \left\{ \frac{1+x}{1+x+x^3} - \frac{1-x^2}{1+x^3} \right\}.$$

$$23. \frac{a^3}{(a+b)(a+c)(x-a)} + \frac{b^3}{(a+b)(b-c)(x+b)} - \\ \frac{c^3}{(a+c)(b-c)(x+c)}.$$

$$24. \left\{ \frac{9x^2}{a^2} - \frac{24x}{a} + 16 - \frac{25a^2}{x^2} \right\} \div \left\{ \frac{3x}{a^2} - \frac{4}{a} - \frac{5}{x} \right\}.$$

$$25. \left\{ x^2 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) \right\} \times \left\{ \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^2(x^2-1)} \right\}^3.$$

$$26. \frac{\left(\frac{a+b}{a-b}\right)^2 + \left(\frac{a-b}{a+b}\right)^2 - 2}{\left(\frac{a+b}{a-b}\right)^3 + \left(\frac{a-b}{a+b}\right)^3 + 2}.$$

$$27. \frac{b^3}{(a+b)^3} + \frac{a^3}{(a-b)^3} - \frac{2b}{a+b} - \frac{2a}{a-b} + 2.$$

$$28. \frac{x^2}{(x-y)(a+x)} + \frac{y^2 - 2xy}{(x-y)(a+y)} + \frac{y^2}{(a+y)^2}.$$

$$29. \frac{\frac{x}{y} + \frac{y}{x} - 1}{\frac{x^2}{y^2} - 1} \times \frac{\frac{x}{y} + \frac{y}{x} + 1}{\frac{x^2}{y^2} + 1} \times \left(\frac{x^4}{y^4} - \frac{x^2}{y^2} \right).$$

$$30. \left\{ \frac{1}{x} - \frac{1}{x + \frac{1}{y + \frac{1}{z}}} \right\} \left(y + \frac{1}{x} + \frac{1}{z} \right).$$

Show that:

$$31. \left(\frac{1}{a} + \frac{1}{b}\right)(a^3 + b^3 - c^3) + \left(\frac{1}{b} + \frac{1}{c}\right)(b^3 + c^3 - a^3) \\ + \left(\frac{1}{c} + \frac{1}{a}\right)(c^3 + a^3 - b^3) = 2(a + b + c).$$

$$32. \frac{a+x}{x(x-y)(x-z)} + \frac{a+y}{y(y-x)(y-z)} + \frac{a+z}{z(z-x)(z-y)} = \frac{a}{xyz}.$$

$$33. \left(\frac{a+x}{a-x}\right)^2 - \frac{4x}{ab} = 1, \text{ when } x = a(1 \pm \sqrt{b}).$$

$$34. \left(\frac{2x+a+c}{2x+b+c}\right)^2 - \frac{x+a}{x+b} = 0, \text{ when } x = \frac{c^2 - ab}{a+b-2c}.$$

$$35. \left(\frac{x+a}{x+b}\right)^3 - \frac{x+2a+b}{x-a-2b} = 0, \text{ when } x = \frac{b-a}{2}.$$

$$36. mx + ny = p, \text{ when } x = \frac{lp - nq}{lm - kn}, y = \frac{mq - kp}{lm - kn}.$$

$$37. (x-a+b)^2 - (x-a)(a-b) = 0, \text{ when } x = \frac{a^2 + ab + b^2}{a+b}.$$

$$38. \frac{12x+10a}{3x+a} + \frac{117a+28x}{9a+2x} = 18, \text{ when } x = 3a.$$

$$39. \frac{a}{\sqrt{x}-\sqrt{b}} - \frac{b}{\sqrt{x}-\sqrt{a}} = \sqrt{a}-\sqrt{b}, \text{ when } x = (\sqrt{a}+\sqrt{b})^2.$$

$$40. \sqrt{1-\sqrt{1-x}} = \frac{1-b}{1+b}, \text{ when } x = 2\left(\frac{1-b}{1+b}\right)^2 - \left(\frac{1-b}{1+b}\right)^4.$$

$$41. \frac{a^2}{(a-b)(a-c)(a-d)} + \frac{b^2}{(b-a)(b-c)(b-d)} \\ + \frac{c^2}{(c-a)(c-b)(c-d)} + \frac{d^2}{(d-a)(d-b)(d-c)} = 0.$$

$$42. \frac{x-2y}{x+2y} + \frac{x-2z}{x+2z} = 1,$$

$$\text{when } x = \frac{12yz}{\sqrt{(y+z)^2 + 12yz} - (y+z)}.$$

CHAPTER VI.

Simple Equations.

57. When two algebraical expressions are connected by the sign ($=$), they are said to form an equation.

When the equality is such that it is true for *all* values of the letters in the given expressions, it is called an *identity*.

Thus, $(x + a)(x + b) = x^2 + (a + b)x + ab$
 and $(a + b)^2 - (a - b)^2 = 4ab$ } are identities.

58. When the condition of equality is such that some one or other of the letters must have particular values or a limited number of values, the statement of equality is termed an *equation of condition*, or, more briefly, an *equation*.

Thus, it may be found on trial that the equality

$$4x + 2 = 3x + 5$$

is true only when $x = 3$. Such an expression is therefore an equation.

59. The letters of an equation to which particular or a limited number of values must be given are termed *unknown* quantities.

Equations may contain *one, two, three*, or more unknown quantities.

The determination of the particular value or values of the unknown quantities is called the *solution* of the equation, and each of the values which *satisfies* the equation is said to be a *root* of the equation.

60. The expressions on the left and right sides of the sign ($=$) are termed the first and second sides respectively. It follows, therefore, that—

1. *If both sides of an equation be multiplied by the same quantity, the equation still subsists.*

2. *If both sides be divided by the same quantity, the equation still holds.*

3. *Any term may be transposed from one side to the other if the sign of the term be changed.*

Thus, if $3x + a = b$, we must have also

$$3x = b - a,$$

for this results from subtracting a from each side of the equation.

4. *The equation holds if every term on both sides has its sign changed.*

Thus, if $ax + b = cx - d$, we may reason as follows:—

The quantity $(ax + b)$ looked upon as a whole is given equal to the quantity $(cx - d)$ looked upon as a whole. If we change the *qualities* of these quantities, they will evidently be still equal.

$$\text{Hence, } -(ax + b) = -(cx - d) \\ \text{or, } -ax - b = -cx + d.$$

Now, this is the result of changing the sign of every term on both sides of the given equation.

5. *The sides of an equation may be reversed without destroying the equality.*

Thus, if $mx + n = px + q$, it must also follow that

$$px + q = mx + n.$$

6. *The sides of an equation may be raised to the SAME POWER, or we may extract the same root of both sides, and the equation still subsists.*

61. Simple equations are those in which the unknown quantities are not higher than the first degree, when the equations are reduced to a rational integral form.

The following is the general method adopted in solving a simple equation involving only one unknown quantity—

1. *Clear of fractions if necessary.*

2. *Transpose all the terms involving the unknown quantity to the first side of the equation, and all the remaining terms to the second side.*

3. *Simplify both sides if necessary, and divide both sides by the coefficient of the unknown quantity.*

Ex. 1. Solve the equation $5x + 6 = 3x + 12$.

Transposing the terms, we have—

$$5x - 3x = 12 - 6.$$

Now, simplifying, we get—

$$2x = 6;$$

and dividing each side by the coefficient of the unknown quantity, viz., by 2, we have—

$$x = 6 \div 2 = 3.$$

VERIFICATION.—Putting the value 3 for x in each side of the given equation, the first side becomes $5 \times 3 + 6$ or 21; and the second side becomes $3 \times 3 + 12$ or 21. The value of x found therefore *satisfies* the given equation.

Ex. 2. Given $\frac{x-2}{2} + \frac{x}{3} = 20 - \frac{x-6}{2}$, find x .

Clearing of fractions, by multiplying *every term* on each side by the L.C.M. of the denominators, viz., by 6, we get—

$$3(x-2) + 2x = 20 \times 6 - 3(x-6).$$

(Beginners often neglect to multiply integral terms such as 20 by the L.C.M.)

$$\text{or } 3x - 6 + 2x = 120 - 3x + 18, \text{ or, transposing,}$$

$$3x + 2x + 3x = 120 + 18 - 6, \text{ or, simplifying,}$$

$$8x = 144,$$

or dividing each side by 8, the coefficient of x , we have—

$$x = 18, \text{ the value required.}$$

(It will be good practice for the student to verify this result as in the last example).

Ex. 3. $\frac{4x-21}{7} + 7\frac{1}{2} + \frac{7x-28}{3} = x + 3\frac{1}{2} - \frac{9-7x}{8} + 1\frac{1}{2}.$

It is sometimes convenient to first *partially* clear off fractions. Thus, multiplying each side by 72, we have—

$$\frac{72(4x-21)}{7} + 47 \times 12 + 24(7x-28)$$

$$= 72x + 15 \times 18 - 9(9-7x) + 6;$$

$$\text{or } \frac{288x}{7} - 72 \times 3 + 564 + 168x - 672,$$

$$= 72x + 270 - 81 + 63x + 6;$$

or, transposing,

$$41\frac{1}{2}x + 168x - 72x - 63x$$

$$= 270 - 81 + 6 + 216 - 564 + 672;$$

or, simplifying,

$$74\frac{1}{2}x = 519; \text{ or, multiplying each side by 7,}$$

$$519x = 519 \times 7;$$

$$\therefore x = 7.$$

Ex. XV.

$$1. 5x + 2 = 2x + 11.$$

$$2. \frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 2\frac{1}{2}.$$

$$3. 2x + a = 3x - b.$$

$$4. 3(x-7) + 4x = 2(2x-4) + 2.$$

$$5. \frac{x-1}{2} + \frac{x+3}{4} = \frac{2x-7}{6} + \frac{8x-1}{12}.$$

$$6. \frac{7x-8}{13} + \frac{3x-2}{7} = \frac{10x+3}{11} - \frac{7x-3}{18}.$$

$$7. \frac{4x-15}{9} - \frac{2x+3}{6} = \frac{5x-1}{12} - 3\frac{1}{2}.$$

$$8. \frac{8(5x+2)}{3} - \frac{2x-1}{8} = \frac{17x-2}{4} + \frac{5\frac{3}{8} + 80x}{7}.$$

$$9. ax + bc = bx + ac.$$

$$10. \frac{x}{a} + \frac{a}{b} = \frac{x}{b} + \frac{b}{a}.$$

$$11. \frac{x-a}{b} + \frac{x-b}{a} = 2 + \frac{cx-a^2}{ab}.$$

$$12. abx + b^2 = b^2x + a^2.$$

$$13. \frac{x}{a} + \frac{x}{b} + \frac{x}{c} = ab + ac + bc.$$

$$14. \frac{x+b}{a} = \frac{a+x}{b}.$$

$$15. \frac{ax + bx + cx}{abc} = a + b + c.$$

$$16. x - \frac{a^2 - 3bx}{a^2} - b^2 = \frac{bx}{a} + \frac{6bx - 5a^2}{2a^2} - \frac{bx + 4a}{4a}.$$

$$17. \cdot 15x + \cdot 025 = \cdot 075x + \cdot 175.$$

$$18. \frac{3 - \cdot 125x}{8} + \cdot 16 = \frac{2 + \cdot 1875x}{7} - \cdot 083.$$

$$19. \frac{1}{2x} + \frac{1}{12x} = \frac{1}{6x} + 1\frac{1}{2}.$$

$$20. (x+a)(x+b) = (x+c)(x+d).$$

$$21. (x-a)(x-b) = (x-\frac{a+b}{2})^2.$$

$$22. \frac{1}{ax} + \frac{1}{bx} + \frac{1}{cx} = \frac{1}{2} (a+b+c)^2 - \frac{1}{2} \left(\frac{a}{bax} + \frac{b}{acx} + \frac{c}{abx} \right).$$

$$23. \frac{x-a}{bc} + \frac{x-b}{ac} + \frac{x-c}{ab} = 2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

$$24. \frac{1-ax}{bc} + \frac{1-bx}{ac} + \frac{1-cx}{ab} = \left(\frac{2}{a} + \frac{2}{b} + \frac{2}{c} \right)x.$$

Abbreviated Methods for Particular Cases.

62. When the unknown quantity is involved in both numerator and denominator, it is often convenient to reduce such fractions to *mixed numbers*.

$$\text{Ex. Solve the equation } \frac{6x-7}{x+2} - \frac{12x+18}{3x-5} = 2.$$

$$\text{By division we get } \frac{6x-7}{x+2} = 6 - \frac{19}{x+2};$$

$$\text{and } \frac{12x+18}{3x-5} = 4 + \frac{38}{3x-5}.$$

Hence the given equation becomes—

$$\left(6 - \frac{19}{x+2} \right) - \left(4 + \frac{38}{3x-5} \right) = 2;$$

$$\text{or } 6 - \frac{19}{x+2} - 4 - \frac{38}{3x-5} = 2;$$

$$\text{or, transposing, } -\frac{19}{x+2} = \frac{38}{3x-5} + 2 - 6 + 4;$$

$$\text{or } -\frac{19}{x+2} = \frac{38}{3x-5};$$

or, dividing each side by -19, we have—

$$\frac{1}{x+2} = -\frac{2}{3x-5}.$$

Hence, multiplying each side by the L.C.M. of the denominators—

$$\begin{aligned} 3x - 5 &= -2(x + 2) = -2x - 4, \\ \text{or } 3x + 2x &= -4 + 5, \\ \text{or } 5x &= 1, \\ \therefore x &= \frac{1}{5}. \end{aligned}$$

63. When each side of an equation consists solely of a single fraction, the numerator of either fraction may change places with the denominator of the other.

Let $\frac{a}{b} = \frac{p}{q}$ be the equation.

Multiply each side by b , then, by Art. 60 (1.)—

$$\frac{a}{b} \times b = \frac{p}{q} \times b, \text{ or } a = \frac{pb}{q}.$$

Divide each side by p , then, by Art. 60 (2.)—

$$\frac{a}{p} = \frac{pb}{q} \div p, \text{ or } \frac{a}{p} = \frac{b}{q}.$$

Here the denominator b of the first side of the given equation has changed places with the numerator p of the second side.

And similarly we may show that $\frac{q}{b} = \frac{p}{a}$, where the other numerator and denominator have changed places.

COR. The two sides of an equation of the form $\frac{a}{b} = \frac{p}{q}$ may be inverted.

For interchanging p and b in the last result, viz., $\frac{q}{b} = \frac{p}{a}$,

we get $\frac{q}{p} = \frac{b}{a}$, and therefore also, by Art. 60 (5.), we have $\frac{b}{a} = \frac{q}{p}$.

(The student is cautioned against inverting the separate terms of the two sides of an equation when there are more than one term on each side.)

64. When each side of an equation consists solely of a single fraction, we may perform the following operations:—

1. We may add or subtract the numerator and denominator of EACH fraction for a new numerator or denominator, and retain either the original numerator or denominator for the other term of the fraction, both sides being always similarly treated.

Thus, if $\frac{a}{b} = \frac{p}{q}$, we have—

$$(i.) \frac{a+b}{b} = \frac{p+q}{q}, \quad (ii.) \frac{a-b}{b} = \frac{p-q}{q},$$

$$(iii.) \frac{a+b}{a} = \frac{p+q}{p}, \quad (iv.) \frac{a-b}{a} = \frac{p-q}{p},$$

or, (v.) we may have equations formed by inverting each of these.

These results are easily obtained—

For, since $\frac{a}{b} = \frac{p}{q}$, we have, adding unity to each side—

$$\frac{a}{b} + 1 = \frac{p}{q} + 1 \text{ or } \frac{a+b}{b} = \frac{p+q}{q}.$$

And so, by subtracting unity from each side, we get—

$$\frac{a-b}{b} = \frac{p-q}{q}, \text{ and so on.}$$

2. We may take the SUMS of the numerator and denominator of each for new numerators or denominators, and the DIFFERENCES for the other terms of the fraction; and VICE VERSA, both sides being always similarly treated.

$$\text{Thus, if } \frac{a}{b} = \frac{c}{d} \text{ we have also } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\text{and } \frac{a-b}{a+b} = \frac{c-d}{c+d},$$

for we have just shown that $\frac{a+b}{b} = \frac{p+q}{q}$,

$$\text{and } \frac{a-b}{b} = \frac{p-q}{q}.$$

Hence, dividing equals by equals, we get—

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{p+q}{q} \div \frac{p-q}{q},$$

or $\frac{a+b}{a-b} = \frac{p+q}{p-q}$ which is the first result.

And inverting each side, we have, by Art. 63 (Cor.)—

$$\frac{a-b}{a+b} = \frac{p-q}{p+q}$$

Ex. 1. Solve the equation $\frac{mx+a+b}{nx-c-d} = \frac{mx+a+c}{nx-b-d}$.

By Art. 63, we have $\frac{mx+a+b}{mx+a+c} = \frac{nx-c-d}{nx-b-d}$

Then, by Art. 64 (1.), retaining the numerators and taking the *differences* for new denominators, we have—

$$\frac{mx+a+b}{b-c} = \frac{nx-c-d}{b-c};$$

or, multiplying each side by $(b-c)$ —

$$mx+a+b = nx-c-d; \text{ or, transposing—}$$

$$(m-n)x = -(a+b+c+d); \therefore x = -\frac{a+b+c+d}{m-n}.$$

Ex. 2. Solve $\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = a$.

We may consider the quantity a as a fraction whose denominator is *unity*, or as $\frac{a}{1}$.

Then, Art. 64 (2.), taking the *sum* and *difference*, we have—

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or—}$$

$$\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or, squaring—}$$

$$\frac{a+x}{a-x} = \frac{a^2+2a+1}{a^2-2a+1}.$$

Again, taking the *difference* and *sum*, we have—

$$\frac{2x}{2a} = \frac{4a}{2a^2 + 2}; \text{ or—}$$

$$\frac{x}{a} = \frac{2a}{a^2 + 1}.$$

$$\therefore x = \frac{2a^2}{a^2 + 1};$$

65. We now give an example to show that sometimes the easy solution depends on an advantageous arrangement of the terms on the two sides of the equation.

Ex. Solve $\sqrt{x+4} + \sqrt{x-3} = 7$.

Transposing, we have—

$$\sqrt{x+4} = 7 - \sqrt{x-3}; \text{ squaring, then—}$$

$$x+4 = 49 - 14\sqrt{x-3} + (x-3);$$

subtracting x from each side and transposing, then—

$$14\sqrt{x-3} = 49 - 3 - 4 = 42.$$

$$\therefore \sqrt{x-3} = 3; \text{ or squaring,}$$

$$x-3 = 9; \therefore x = 3 + 9 = 12.$$

Should the student commence by squaring at once, he will render the equation more complicated.

Ex. XVI.

$$1. \frac{3x+7}{x+4} = \frac{3x-13}{x-4}.$$

$$2. (x-a)(x-b) = (x-c)(x-d).$$

$$3. \frac{3x+13}{15} - \frac{3x+10}{5x-50} = \frac{x}{5}.$$

$$4. \frac{1-25x}{15} - \frac{3-2\frac{1}{2}x}{14(x-1)} = \frac{28-5x}{3} - \frac{10x-11}{30} + \frac{x}{3}.$$

$$5. \frac{x-4}{6x+5} + \frac{3x-13}{18x-6} = \frac{1}{3}.$$

$$6. \frac{3-2x}{1-2x} - \frac{5-2x}{7-2x} = 1 - \frac{4x^2-2}{7-16x+4x^2}.$$

$$7. \frac{18x - 22}{13 - 2x} + 6x + \frac{1 + 16x}{8} = 13\frac{1}{4} - \frac{101 - 64x}{8}.$$

$$8. \frac{6x + 5}{3x + 1} + \frac{58\frac{1}{2} + 14x}{9 + 2x} = 9.$$

$$9. \frac{3}{4x - 9} + \frac{1\frac{1}{2}}{6x - 21} = \frac{1}{x - 2\frac{1}{2}}.$$

$$10. \frac{4x - 7}{2x - 9} + \frac{2 - 14x}{7} + \frac{3\frac{1}{2} + x}{14} = \frac{10 - 3\frac{1}{2}x}{2} - \frac{19}{21}.$$

$$11. \frac{6x^2 - 7x - 6\frac{2}{3}}{2x - 3} - \frac{9x^2 - 12x - 19}{3x - 5} = \frac{17}{6x - 7}.$$

$$12. \frac{ax + m + 1}{ax + m - 1} + \frac{ax + n}{ax + n - 2} = \frac{ax + m}{ax + m - 2} +$$

$$\frac{ax + n + 1}{ax + n - 1}.$$

$$13. \frac{\sqrt{x - a + b} - \sqrt{x + a - b}}{\sqrt{x - a + b} + \sqrt{x + a - b}} = \frac{a - b}{a + b}.$$

$$14. \frac{1 - \sqrt{1 - \sqrt{1 - x}}}{1 + \sqrt{1 - \sqrt{1 - x}}} = b.$$

$$15. \sqrt{2x + 10} + \sqrt{2x - 2} = 6.$$

$$16. \sqrt{8 - x} - \frac{3}{\sqrt{1 - x}} = \sqrt{1 - x}.$$

$$17. \sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}} = 2.$$

$$18. \frac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = b.$$

$$19. \frac{a}{\sqrt{x} - \sqrt{b}} - \frac{b}{\sqrt{x} - \sqrt{a}} = \frac{a - b}{\sqrt{x}}.$$

$$20. \frac{a}{\sqrt{x} + \sqrt{a}} = \frac{a}{\sqrt{x} - \sqrt{a}} + \sqrt{a}.$$

$$21. \sqrt{\frac{\sqrt{x + a}}{\sqrt{x - a}}} - \sqrt{\frac{\sqrt{x - a}}{\sqrt{x + a}}} = \sqrt{x - a^2}.$$

$$22. \frac{1}{x-2} - \frac{2 + \frac{1}{2}x^2 - \frac{1}{2}x^2}{6-5x+x^2} - \frac{1}{2}x = \frac{5}{x-3}$$

$$23. \frac{4x+5}{x+1} + \frac{x+5}{x+4} = \frac{2x+5}{x+2} - \frac{x^2-10}{x+3} + x$$

$$24. \frac{ax+b}{cx+d} + \frac{cx+e}{ax+f} = \frac{a^2+c^2}{ac}$$

$$25. \text{ If } a + \sqrt{a^2 + x^2} = b + x, \text{ then } x = \frac{b}{2} \left(\frac{a}{b-a} - 1 \right).$$

$$26. \text{ If } \sqrt{x} - \sqrt{a} - \sqrt{ax+x^2} = \frac{1}{2}\sqrt{a}, \text{ then } x = \frac{1}{4}a.$$

$$27. \text{ If } \frac{1+x}{1-x} = \frac{b}{a} \cdot \frac{1+x+x^2}{1-x+x^2}, \text{ then } x = \sqrt[3]{\frac{b-a}{b+a}}.$$

$$28. \text{ If } \frac{a+c(a-x)}{a+c(a+x)} = \frac{a}{a+2cx} + \frac{a-x}{x}, \text{ then } x = a.$$

$$29. \text{ If } \frac{a+x}{\sqrt{2ax+x^2}} = \frac{b+1}{b-1}, \text{ then } x = \frac{a}{2\sqrt{b}} (\sqrt{b}-1)^2.$$

$$30. \text{ If } \frac{a(1+b)+bx}{a(1+b)-bx} = \frac{a}{a-2cx} - \left(\frac{a}{x} + 1 \right), \text{ then } x = -a.$$

$$31. \text{ If } \frac{x^2-3x-9}{x-5} + \frac{x^2-7x-17}{x-9} \\ = \frac{x^2-6x-15}{x-8} + \frac{x^2-4x-11}{x-6}, \text{ then } x = 7.$$

$$32. \text{ If } \sqrt{x+m} + \sqrt{x+n} = \sqrt{m} + \sqrt{n}, \text{ then } x = 0.$$

$$33. \text{ If } \frac{x^2-17}{x-4} + \frac{2x^2+7x-13}{2x-3} \\ = \frac{2x^2+x-30}{2x-7} + \frac{x^2+4x-4}{x-1}, \text{ then } x = \frac{5}{2}.$$

$$34. \text{ If } \left(\frac{x+.5}{x+.7} \right)^{\frac{1}{2}} + \frac{.1}{x+.6} = 1, \text{ then } x = -.7.$$

$$35. \text{ If } \sqrt{x+3.2} - \sqrt{x+.8} = 2\sqrt{x-1.6}, \text{ then } x = 1.7.$$

36. If $\frac{2.7x - 1}{\sqrt{2.7x} + 1} = \frac{\sqrt{2.7x} + 7}{2}$, then $x = 30$.

37. If $\sqrt{4x + .01} + \sqrt{4x - .36} = 3.7$, then $x = .9$.

38. If $\left(\frac{m-x}{m+x}\right)^2 = 1 - \frac{nx}{mp}$, then $x = m\left(\pm 2\sqrt{\frac{p}{n}} - 1\right)$.

Problems producing Simple Equations involving One Unknown Quantity.

66. To solve an algebraical problem we represent the required or unknown quantity by a letter, as x , and then express the given conditions in algebraical language. Thus we form an equation, the solution of which gives the required value of the unknown quantity.

Ex. 1. My purse and money are together worth 24 shillings, and the money is worth seven times the purse. Find the value of each.

Let x = the value in shillings of the purse,

Then $7x$ = " " " money.

Now, by problem the value of both together is 24 shillings.

Hence we have—

$$x + 7x = 24$$

$$\text{or } 8x = 24$$

$\therefore x = 3$, the value in shillings of the purse,
and $\therefore$ also $7x = 7 \times 3 = 21$, " money.

Ex. 2. What number is that to which, if 36 be added, the sum shall be equal to 3 times the number?

Let x = the number;

$\therefore x + 36$ = the sum when 36 is added,

and $3x = 3$ times the number.

Hence, by problem—

$$x + 36 = 3x,$$

$$\text{or } x - 3x = -36,$$

$$\text{or } -2x = -36;$$

$$\therefore x = \frac{-36}{-2} = 18, \text{ the number required.}$$

Ex. 3. The distance between two towns is such that a train, whose speed is 30 miles an hour, takes 1 hour more in going 10 miles over 5 times the distance than a train whose speed is 20 miles an hour takes in going within 4 miles of 3 times the distance. Find the distance between the towns.

Let x = the distance required in miles.

Then $5x + 10$ = 5 times the distance together with 10 miles,
and $\frac{5x + 10}{30}$ = time in hours to travel this distance at 30 miles an hour.

And so, $\frac{3x - 4}{20}$ = time in hours to travel 4 miles less than 3 times the required distance, at 20 miles an hour.

But by the problem the former of these times exceed the latter by 1 hour:

$$\text{Hence } \frac{5x + 10}{30} - \frac{3x - 4}{20} = 1.$$

From this we easily find $x = 28$.

Hence 28 miles is the distance required.

Ex. 4. Find the price of an article, when as many can be bought for 1s. 4d. as can be bought for 2s. after the price has been raised 1d.

Let x = the price required in *pence*;

Then $\frac{16}{x}$ = number of articles bought for 1s. 4d.

And $x + 1$ = the raised price in *pence*.

$\therefore \frac{24}{x + 1}$ = number of articles bought for 2s. at the raised price.

But, by the problem, the number of articles in each case is the same.

$$\text{Hence } \frac{16}{x} = \frac{24}{x + 1}, \text{ from which } x = 2.$$

Hence 2d. is the price required.

Ex. 5. A man sells geese at as many shillings each as the number he has, and having returned 5s., finds that if he had

had 2 more to sell on the same condition, and had returned 3s., he would have had 38s. more. How many had he?

Let x = the number required;

Then $x^2 - 5$ = number of shillings received.

Also, on the second supposition,

$(x + 2)^2 - 3$ = number of shillings he would have received.

Now, by the problem, this latter number is 38 more than the former.

Hence $(x + 2)^2 - 3 = x^2 - 5 + 38$, from which we find $x = 8$.

Ex. 6. A waterman finds that he can row 5 miles in $\frac{3}{4}$ hour with the tide, and that it takes him $1\frac{1}{2}$ hours to row the same distance against the tide when it is but half as strong. What is the velocity of the tide?

Let x = the velocity of the tide in miles per hour.

Now the velocity of the boat when going *with* the tide
 $= 5 \div \frac{3}{4} = \frac{20}{3}$.

$\therefore \frac{20}{3} - x$ = velocity of the boat when there is no tide.

Again, velocity of boat *against* the tide when it is half as strong $= 5 \div 1\frac{1}{2} = \frac{10}{3}$.

$\therefore \frac{10}{3} + \frac{x}{2}$ = velocity of the boat, when there is no tide.

Hence we have—

$$\begin{aligned} \frac{10}{3} + \frac{x}{2} &= \frac{20}{3} - x; \text{ from which} \\ x &= 2\frac{2}{3}. \end{aligned}$$

$\therefore$ The velocity required is $2\frac{2}{3}$ miles per hour.

Ex. XVII.

1. If I add 25 to 3 times a certain number, I obtain the same result as if I subtract 25 from 8 times the number. Find the number.

2. Divide 70 into 2 such parts that the one shall be as much above half the number as the other is above 15.

3. Divide £720 among A, B, and C, so that B may have twice as much as C, and A as much as B and C together.

4. There are two trains, one of which goes 5 miles an hour faster than the other, and the former performs a journey of 100 miles, while the latter goes 75 miles. Find their respective rates.

5. A horse when let out for hire brings in a *clear gain* of 10s. per day, but costs 1s. 6d. daily for food. At the end of 30 days his master had gained £11. 11s. Required the number of days for which he was hired.

6. A and B have 4 guineas between them, and play at hazard. B loses $\frac{1}{2}$ of his money, and afterwards gains $\frac{1}{3}$ of what he then had. It is then seen that B has as much money as A had at the end of the first game. How much had each at first?

7. A and B have respectively an equal number of florins and crowns. B pays a debt of 4s. to A, and then A's money is just half B's. Find what each had.

8. A workman, instead of adopting the 9 hours' system, worked 10 hours daily, and had a corresponding rise of wages. By this means his wages were increased 4s. weekly. Find his original wages.

9. A person who has regular wages of 26s. weekly, thinking to better himself, takes a job at higher wages. He is, however, put on half-time during 20 weeks of the year, and finds himself at the end of the year £4. 12s. worse off. Required his increased wages.

10. A company of men, arranged in a hollow square 4 deep, numbered 144. What was the number in a side of the square?

11. In an examination paper there were two series of questions, and the questions of the second series carried each 3 marks more than those of the first series. A candidate who attempted 3 of the first series, obtaining half marks for them, and 5 of the second series, obtaining for these full marks, got altogether 80 marks. Find the number of marks attached to each question of the first series.

12. A grocer has tea at 3s. 4d. and at 4s. He sells altogether 64 pounds, thereby realizing £12. How much did he sell of each?

13. If 11 be subtracted from 5 times a certain number, and the remainder divided by 6, the quotient will exceed by 2 the quotient obtained by subtracting 3 from 4 times the number and dividing the remainder by 7. Find the number.

14. A garrison of 1,250 men were provisioned for 64 days; but after 22 days a certain number were called away, and it was found that the remaining provisions lasted the number left for 70 days. Find the number told off.

15. At a railway station £15 was taken for single fares, and £33. 15s. for returns. The number of return tickets exceeded the single tickets by 10, and the price of a return ticket was half as much again as a single ticket. Find the fare for a single journey.

16. In a tour lately made round the world, the distance travelled by water was 20,000 miles, and by land 8,000 miles; and the whole time taken was 220 days. Supposing the rate by water to be two-thirds of that by land, find the number of days travelled by land.

17. The distance between A and B is 32 miles. A person starting from A, at the rate of 4 miles an hour, meets another who started from B half an hour later, at a rate of $3\frac{1}{2}$ miles an hour. At what point will they meet?

18. There are two clocks, one of which gains twice as much per day as the second loses, and they are set right at noon on Monday. When it is noon on Thursday by the first clock, it is 11-50 A.M. by the second. What is the gain per day of the first clock?

19. A draper raises his goods a certain rate per cent., and afterwards reduces them to the original price by lowering them $13\frac{1}{3}$ per cent. Find the original rise per cent.

20. Required the distance between two towns such that a person can perform the journey one hour sooner when he walks 4 miles an hour than when he walks $3\frac{1}{2}$ miles an hour?

21. The sum of £12. 15s. is paid away with an equal number of sovereigns, crowns, and sixpences. Required the number of each.

22. A walks along an inclined plane at a certain rate, and B walks along the base of the plane at a rate of one-third of

a mile per hour less than A. The inclination of the plane is such that A is always vertically over B, and that at the end of half an hour they are exactly five-sixths of a mile apart. Find the respective rates of A and B.

23. There is a direct road over a hill between two stations at the foot of each side. The distance on the one side from the foot to the top is 5 miles, and the road down the other side forms a right angle with the road up. It is also known to be 1 mile less down the hill than the direct distance by tunnel between the two stations. Find the distance down the hill.

24. Two trains, whose respective lengths are 122 and 98 yards, and the former of which is going at the rate of 35 miles an hour, pass each other in 30 seconds. Find the rate and relative direction of the second train.

25. A man bought a number of sheep for £132, and having lost 10, and sold 20 that were diseased at 6s. per head below cost price, disposed of the remainder for £116, thereby realizing his outlay. How many did he buy?

26. A boy spends 10s. in oranges and apples. The oranges were bought at 5 for 6d., and the apples at 3 for 2d.; and their number together amounted to 132. What did he spend on each?

27. If B is allowed 2 hours more time than A takes to do a piece of work, he will do 4 times as much, and if C is allowed 1 hour more than A, he also can do 4 times as much. Moreover, D requires 4 hours more than A to do the piece of work. Also, the work done by A and B together is the same as that done by C and D together in the same time. Required the respective times for A, B, C, D to do a piece of work.

28. A person sells out £1,200 Three and a Half per Cent. stock, and invests the money in Two and a Half per Cents., whose price is 14 lower than the first-named stock. The loss in annual income is £7. Find the price of the first-named stock.

29. The banker's discount on a certain sum of money at 5 per cent. per annum is equal to the true discount on a sum £50 larger. Find the sum.

30. An express train, which ought to perform its journey in $2\frac{1}{2}$ hours, after having gone uniformly 80 miles, finds itself 6 minutes behind. However, by increasing the speed to as many miles per hour as there were miles in half the journey, it just arrived at its destination in time. Find the original speed of the train, and the length of the journey.

31. A vessel contains a quantity of spirit (sp. gr. .9) and water, and a cylinder of wood (sp. gr. .92), whose length is 10 inches, floats upright, so as to be just covered by the spirit. Find how much of the cylinder floats in the water.

32. A mixture of hydrogen and oxygen is found to condense when fired, to 16 vols. of steam. Now, every 3 vols. of such a mixture is known to condense to 2 vols., when the original gases are in the proportion of 2 : 1. Find the quantity of each.

33. A mixture of 100 grams of sodic and potassic sulphates yielded a gram of baric sulphate. Now, each gram of sodic sulphate yields b grams of baric sulphate, and each gram of potassic sulphate yields c grams of baric sulphate. Find the amount of sodic and potassic sulphates in the mixture.

34. If a oxen consume b acres of grass in c weeks, and a' oxen consume b' acres of grass in c' weeks, the grass growing uniformly, find the week's growth of an acre.

35. The freezing and boiling points of a common thermometer are marked 32° and 212° respectively; those on the centigrade thermometer are marked 0° and 100° . At what temperature do the graduations agree?

36. A person going at the rate of a miles per hour finds himself b hours too late when he has c miles farther to go. How much must he increase his speed to reach home in time?

Simultaneous Equations of the First Degree with two Unknown Quantities.

67. Suppose we have given the equation $3x - 4y = 5$, then by ascribing to y a series of values we get a corresponding series of values for x .

Thus we may have $\left. \begin{matrix} x = 3 \\ y = 1 \end{matrix} \right\}$, $\left. \begin{matrix} x = 7 \\ y = 4 \end{matrix} \right\}$, $\left. \begin{matrix} x = 11 \\ y = 7 \end{matrix} \right\}$, &c.

Again, if another equation, as $4x + y = 32$, be given, we may in the same way obtain a series of pairs of values which satisfy it. And, further, if the two equations are *distinct* and *compatible*, there is always a *pair of values common to the two equations*. This pair of values then satisfies both equations, and the equations are called **simultaneous equations**.

The methods of solving simultaneous equations will now be explained.

FIRST METHOD.—*Equalize the coefficients of one of the unknown quantities in both equations, and add or subtract the equations so obtained, so as to obtain an equation with one unknown quantity.*

$$\begin{array}{rcl} \text{Ex. } 4x + 3y = 17 & \dots\dots\dots & (1) \\ 9x - 5y = 3 & \dots\dots\dots & (2) \end{array} \left. \vphantom{\begin{array}{rcl} \text{Ex. } 4x + 3y = 17 \\ 9x - 5y = 3 \end{array}} \right\}.$$

From (1), multiplying each side by 9, we get—

$$36x + 27y = 153 \dots\dots\dots (3).$$

And from (2), multiplying each side by 4, we get—

$$36x - 20y = 12 \dots\dots\dots (4).$$

$$(3) - (4), \text{ then } 47y = 141$$

$$\therefore y = 3.$$

Hence, substituting in (1), we have—

$$4x + 3 \times 3 = 17, \text{ from which we get—} \\ x = 2.$$

Hence, the solution required is $x = 2, y = 3$.

SECOND METHOD.—*Express one of the unknown quantities in terms of the other by means of either equation, and substitute its value in the other.*

Taking the same example, we have—

$$\begin{array}{rcl} 4x + 3y = 17 & \dots\dots\dots & (1) \\ 9x - 5y = 3 & \dots\dots\dots & (2) \end{array} \left. \vphantom{\begin{array}{rcl} 4x + 3y = 17 \\ 9x - 5y = 3 \end{array}} \right\}.$$

$$\text{From (1) we have } 4x = 17 - 3y, \text{ or } x = \frac{17 - 3y}{4} \dots\dots (3).$$

Substituting this value of x in (2) we have—

$$9 \left(\frac{17 - 3y}{4} \right) - 5y = 3,$$

$$\text{or, } 153 - 27y - 20y = 12, \text{ from which—} \\ y = 3.$$

Then from (3) we get $x = \frac{17 - 3 \times 3}{4} = 2$.

THIRD METHOD.—Express one of the unknown quantities in terms of the other by means of each equation, and equate the expression.

We will this time express in each case y in terms of x .

$$\left. \begin{array}{l} \text{We have } 4x + 3y = 17 \dots\dots\dots(1) \\ \quad \quad 9x - 5y = 3 \dots\dots\dots(2) \end{array} \right\}.$$

From (1) $3y = 17 - 4x$, or $y = \frac{17 - 4x}{3} \dots(3)$.

And from (2) $5y = 9x - 3$, or $y = \frac{9x - 3}{5} \dots\dots(4)$.

Equating (3) and (4), then $\frac{17 - 4x}{3} = \frac{9x - 3}{5}$, from which

$$x = 2.$$

Then from (3) by substitution, $y = \frac{17 - 4 \times 2}{3} = 3$.

Simultaneous Equations of the First Degree of Three or more Unknown Quantities.

68. In the case of *three* unknown quantities we may obtain, from the three given equations, two equations with two unknown quantities, and then, by a similar method, from the two obtain an equation with one unknown quantity; and a like method may be pursued for more than three unknown quantities.

$$\begin{array}{lcl} \text{Ex.} & 3x + y + 4z = 25 \dots\dots\dots(1) \\ & 4x + 3y - 5z = -3 \dots\dots\dots(2) \\ & 6x + 7y - 8z = 1 \dots\dots\dots(3) \\ \text{From (1)} & 12x + 4y + 16z = 100 \dots\dots\dots(4) \\ \text{And from (2)} & 12x + 9y - 15z = -9 \dots\dots\dots(5) \\ (5) - (4), \text{ then} & 5y - 31z = -109 \dots\dots\dots(6) \\ \text{Again, from (1)} & 6x + 2y + 8z = 50 \dots\dots\dots(7) \\ (3) - (7), \text{ then} & 5y - 16z = -49 \dots\dots\dots(8) \\ (8) - (6), \text{ then} & 15z = 60 \\ & \therefore z = 4 \end{array}$$

Hence from (8), by substitution—

$$5y - 16 \times 4 = -49,$$

$$\text{from which } y = 3.$$

And hence from (1), by substitution of the known values of y and z —

$$3x + 3 + 4 \times 4 = 25,$$

$$\text{from which } x = 2.$$

Hence the respective values of x, y, z , are 2, 3, 4.

Ex. XVIII.

1. $6x + y = 22, 5x + 3y = 27.$
2. $4x - 3y = 14, 6x + 5y = 40.$
3. $3x + 5y = 44, y - x = 4.$
4. $\frac{x}{4} + \frac{y}{3} = 2\frac{1}{2}, \frac{3x}{5} + \frac{y}{10} = 2\frac{1}{5}.$
5. $\frac{x}{6} + \frac{y}{4} = 4, \frac{x}{4} + \frac{y}{6} = 4\frac{1}{2}.$
6. $\frac{x}{8} + \frac{y}{6} = 1\frac{1}{2}, x + \frac{y}{3} = 4.$
7. $3x + 4y + z = 14, 2x + y + 5z = 19,$
 $5x + 2y + 3z = 18.$
8. $7x + 2y + 3z = 20, 3x - 4y + 2z = 1,$
 $5y - 2x + 7z = 29.$
9. $2x - 5y = 3, 3y - 2z = -1, 4x + 2z = 20.$
10. $ax + by = c, a_1x + b_1y = c_1.$
11. $x + y = a, y + z = b, x + z = c.$
12. $ax + by = d, by + cz = e, ax + cz = f.$
13. $\frac{3x-2y}{4} + \frac{5x+2y}{6} = 5\frac{1}{2}, \frac{x+2y}{7} - \frac{3x-2y}{2} = 1.$
14. $\frac{x+y}{10} + \frac{x-y}{15} = \frac{23}{30}, \frac{x+y}{15} + \frac{x-y}{10} = \frac{7}{30}.$
15. $\frac{3}{x} + \frac{9}{y} = 1\frac{1}{2}, \frac{15}{x} - \frac{11}{y} = 2\frac{1}{2}.$

$$16. \frac{4}{x-2} + \frac{6}{y} = 5\frac{1}{2}, \frac{4}{y} - \frac{1}{x-2} = 0.$$

$$17. 4(x+3y) - 3(x+y) = \frac{1}{2}(x+y)(x+3y),$$

$$20. \frac{20}{x+y} + \frac{9}{x+3y} = 3.$$

$$18. 73y - 5x = (x-5y)(x+3y),$$

$$\frac{2}{x-5y} - \frac{5}{x+3y} = \frac{7}{33}.$$

$$19. \frac{1}{x} + \frac{1}{y} = a, \frac{1}{y} + \frac{1}{z} = b, \frac{1}{x} + \frac{1}{z} = c.$$

$$20. 3x - 2y = 0, 4y - 3z = 0, 5z - 6u = 14, 7u - 2x = 3.$$

$$21. 3x - 2y = 5z - 6y = 7x - 4z = 1.$$

$$22. x + y + z = a, y + z + u = b, z + u + x = c, u + x + y = d.$$

$$23. \frac{x}{m} + \frac{y}{n} = a, \frac{y}{n} + \frac{z}{p} = b, \frac{x}{m} + \frac{z}{p} = c.$$

$$24. x + ay + bz = a, y + az + bx = \beta, z + ax + by = \gamma.$$

$$25. x + ay + a^2z + a^3 = 0.$$

$$x + by + b^2z + b^3 = 0.$$

$$x + cy + c^2z + c^3 = 0.$$

$$26. x + ay + a^2z + a^3u + a^4 = 0.$$

$$x + by + b^2z + b^3u + b^4 = 0.$$

$$x + cy + c^2z + c^3u + c^4 = 0.$$

$$x + dy + d^2z + d^3u + d^4 = 0.$$

$$27. x + ay = a.$$

$$y + bz = \beta.$$

$$z + ct = \gamma.$$

$$t + dx = \delta.$$

$$u + ex = \epsilon.$$

$$28. \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = d$$

$$\frac{a'}{x} + \frac{b'}{y} + \frac{c'}{z} = d'$$

$$\frac{a''}{x} + \frac{b''}{y} + \frac{c''}{z} = d''$$

$$29. x_1 + 2x_2 + 3x_3 + \&c. + nx_n = a_1.$$

$$x_2 + 2x_3 + 3x_4 + \&c. + nx_1 = a_2.$$

$$\&c. = \&c.$$

$$x_n + 2x_1 + 3x_2 + \&c. + nx_{n-1} = a_n.$$

$$30. \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{x}{b} + \frac{y}{c} + \frac{z}{a} = \frac{x}{c} + \frac{y}{a} + \frac{z}{b} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Problems producing Simultaneous Equations of the First Degree.

69. There are certain problems which may be solved with much greater facility by the introduction of more than one unknown quantity.

Ex. 1. Nine men and seven women receive together £3. 2s. 8d., and three men receive 10d. more than four women. Find the receipts of each.

Let x and y respectively represent in pence the receipts of a man and woman.

Then, since £3. 2s. 8d. = 752d., expressing the conditions of the problem in algebraical language—

$$\begin{aligned} 9x + 7y &= 752 \\ 3x - 4y &= 10 \end{aligned} \left. \begin{array}{l} \text{Solving these equations,} \\ \text{we find—} \end{array} \right\} \begin{aligned} x &= 54, \\ y &= 38. \end{aligned}$$

Hence, the receipts of a man and woman are respectively 54d. and 38d., or 4s. 6d. and 3s. 2d.

Ex. 2. Find a fraction such that, if we diminish its numerator by 1, it becomes equal to $\frac{1}{7}$; and if we increase its denominator by 1, it becomes equal to $\frac{1}{6}$.

Let $\frac{x}{y}$ be the required fraction.

$$\begin{aligned} \text{Then, by problem, } \frac{x-1}{y} &= \frac{1}{7} \\ \text{and } \frac{x}{y+1} &= \frac{1}{6} \end{aligned} \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{which equations, when} \\ \text{solved, give—} \end{array}$$

$$x = 6, y = 35.$$

$\therefore \frac{6}{35}$ is the fraction required.

Ex. 3. A's money, together with twice B's and thrice C's, amounts to £38; B's money, together with twice C's and thrice A's, to £35; and C's money, together with twice A's and thrice B's, also to £35. Find the money of each.

Let x, y, z be respectively the number of pounds each has. Then we have—

$$\left. \begin{aligned} x + 2y + 3z &= 38 \\ y + 2z + 3x &= 35 \\ z + 2x + 3y &= 35 \end{aligned} \right\}; \text{ from which we get—}$$

$$x = 5, y = 6, z = 7.$$

Hence, £5, £6, £7 are the respective moneys of A, B, and C.

Ex. XIX.

1. A and B engage in play. A puts down half-a-crown to B's florin. They play twenty games, and then it is found that A has won 4s. How many games did each win?

2. There is a number, the sum of whose two digits is 10, and, if 36 be added to the number, the digits change places. Find the number.

3. A grocer has tea at 3s. 4d. a lb., and sugar at 4s. a stone. He sells £2 worth of the two. If he had raised the tea 20 per cent., and lowered the sugar $12\frac{1}{2}$ per cent., and sold the same quantities of each, his profits would have been 1s. 6d. more. Find the quantity sold of each.

4. Eight times the numerator of a certain fraction exceeds three times the denominator by 3, and five times the numerator added to twice the denominator make 29. Find the fraction.

5. If the number of cows in a field were doubled, there would be 65 cows and horses together; but, if the number of horses be doubled, and that of the cows halved, there would be 46. How many are there of each?

6. Thirty shillings are spent in brandy, and 42s. in gin; 19 bottles being purchased in all. Had the 42s. been spent in brandy, and the 30s. in gin, 17 bottles only would have been bought. Find the cost per bottle of each.

7. A fishmonger receives 240 mackerel. He sells a certain number at 4 for a shilling, but the rest being seized as bad fish, and he being fined 10s., finds himself a loser by 9s. Had he sold them at 3 for a shilling, he would have been a gainer by 5s., if 13 more fish had been seized. How many did he sell, and what did he pay for the lot?

8. Three persons invest their money at 3, 4, 5 per cent. interest respectively. The total amount of interest is £38, and the interest of the first and third together is $2\frac{1}{2}$ that of the second; while the total interest would have been £34 had the rates been 5, 4, 3 per cent. respectively. Find the capital of each.

9. A toll-gate keeper receives 8s. 8d. for the toll of a number of horses, oxen, and sheep, the tolls for each being respectively $1\frac{1}{2}$ d. 1d., $\frac{1}{2}$ d. Had there been twice as many sheep and the number of horses diminished accordingly, he would have received 7s. 2d. Had the oxen passed through free, and the tolls for a horse and sheep respectively been 2d. and $\frac{3}{4}$ d., he would have received 9s. $1\frac{1}{2}$ d. Find the number of each.

10. A, B, C start from the same place. B after a quarter of an hour doubles his rate, while C, who falls, after ten minutes diminishes his rate $\frac{1}{4}$ th. At the end of half an hour A is $\frac{1}{4}$ mile before B, and $\frac{1}{2}$ mile before C, and it is observed that the total distance which would have been walked by the three, had they each continued to walk uniformly from the first, is $6\frac{1}{4}$ miles. Find the original rate of each.

11. A, B, C, working 3, 4, 5 hours respectively, can do $2\frac{1}{4}$ pieces of work; if they each work an hour more, they can finish an extra $\frac{1}{4}$ of a piece; and, if C does not work, the other two, working for 1 and 6 hours respectively, can together finish 1 piece. Find the time required for A, B, C to finish separately a piece of work.

12. There are three numbers such that, if the first be increased by 6, and the second diminished by 5, the product of the results is the product of the first two numbers; if the second be increased by 2 and the third diminished by 3, the product of the results is the product of the second and third; and, if the first be increased by 3 and the third diminished by 6, the product is that of the first and third. Find the numbers.

13. A person performed a journey of $22\frac{1}{2}$ miles, partly by carriage at 10 miles an hour, partly by train at 36 miles an hour, and the remainder by walking at 4 miles an hour. He did the whole in 1 hour 50 minutes. Had he walked the

first portion, and performed the last by carriage, it would have taken him 2 hours $30\frac{1}{2}$ minutes. Find the respective distances by carriage, train, and walking.

14. A and B start from two places C and D, distant 28 miles, and it is found that A reaches D 3 hours after they meet. Had the distance between C and D been 35 miles, A would have reached a point 28 miles from C 2 hours after he met B. Find the respective rates of A and B.

15. Three trains—a luggage, ordinary, and express—move along three parallel pairs of rails, the distance between the stations being 120 miles. The first two start from the same station, and the express from the opposite. The luggage train, starting 2 hours first, is overtaken by the ordinary in 2 hours; and the express train, starting 1 hour after the ordinary, meets the luggage in 1 hour $7\frac{1}{2}$ minutes. Had all three started from the same station, the ordinary would have been overtaken in 2 hours. Find the respective rates of the trains.

16. If (a_1, b_1, c_1) , (a_2, b_2, c_2) , (a_3, b_3, c_3) be the respective compositions by weight of three mixtures of three substances, and d_1, d_2, d_3 be the respective prices of the mixtures, find the price per unit of weight of each substance.

17. By alloying two ingots of gold in two given proportions, we form two new ingots of which the fineness of each is known. What is the fineness of each of the given ingots?

18. A group of n persons play as follows:—The first plays with the second and loses $\frac{1}{a}$ of what he has, the second then plays with the third and loses $\frac{1}{a}$ of what he has, the third then with the fourth, losing $\frac{1}{a}$ of what he has, &c., the n th with the first losing $\frac{1}{a}$ of what he has. At the end they each have b . What had each at first?

19. There are three vessels which are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ full. If all the liquid is poured into the first vessel it is found to be $\frac{29}{30}$ full; and the capacity of the third is $\frac{3}{10}$ of the sum of the capacities of the first and second; show that their capacities are as the numbers 6, 4, 3.

20. Two waggoners, A and B, start at the same time from two stations on the same road, going respectively 3 and 4 miles an hour. When B has gone $\frac{1}{2}$ of his journey, he finds he must return to A's station; and on meeting A finds him $21\frac{1}{2}$ miles on his way. Having reached A's station, he again turns and reaches his destination along with A; find the distance between the stations, and the length of A's journey.

CHAPTER VII.

QUADRATIC EQUATIONS.

70. Equations of this class, when reduced to a rational integral form, contain the *square* of the unknown quantity, but no higher powers.

When the equation contains the *square only* of the unknown quantity, and *not the first power*, it is called a *pure quadratic*.

Thus, $x^2 - 25 = 0$, $4x^2 + 10 = 19$, $5x^2 = 180$, are *pure quadratics*. When the equation contains the *square* of the unknown quantity, *as well as the first power*, it is called an *adjected quadratic*.

Thus, $x^2 - 5x = 6$, $x^2 - x - 30 = 0$, $2x^2 + x + 3 = 6$, are *adjected quadratics*.

Pure Quadratics.

71. To solve these, we treat them exactly as we do simple equations, until we obtain the value of the *square* of the unknown quantity; then, taking the square root of each side, we obtain the *value* of the unknown quantity. It will be

seen (Art. 35) that the unknown quantity in a pure quadratic has always two values, *equal* in magnitude, but of *opposite* sign.

Ex. 1. Given $3x^2 + 12 = 687$, find x .

We have $3x^2 = 687 - 12 = 675$, or $x^2 = 225$.

$$\therefore x = \pm 15.$$

Ex. 2. Given $\frac{2x+7}{2x^2-7x} - \frac{2x-7}{2x^2+7x} = \frac{56}{6x^2-99}$, find x .

Bringing the fractions on the first side to a common denominator, we have—

$$\frac{(2x+7)^2 - (2x-7)^2}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{56x}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{1}{4x^2-49} = \frac{1}{6x^2-99}, \text{ clearing of fractions;}$$

then, $6x^2 - 99 = 4x^2 - 49$, from which

$$x^2 = 25$$

$$x = \pm 5.$$

Affected Quadratics.

72. *Solution by completing the square.*

Suppose we have given the equation $x^2 + 2ax = 3a^2$, to find x .

It will be remembered that $(x^2 + 2ax + a^2)$ is a *perfect square*, viz., the square of $(x + a)$. It is evident, then, that the first side of our equation will become a perfect square by the addition of a^2 as a third term.

Adding, then, a^2 to each side of the given equation, we have—

$$\begin{aligned} x^2 + 2ax + a^2 &= 4a^2, \text{ or} \\ (x + a)^2 &= 4a^2. \end{aligned}$$

Taking now the square root of each side, we have—

$$x + a = \pm 2a.$$

$$\therefore x = \pm 2a - a.$$

$$= a \text{ or } -3a.$$

We may remark that the quantity a^2 , added to the expression $x^2 + 2ax$ in order to make it a *perfect square*, is the square of half the coefficient of x . The operation itself is called *completing the square*.

An adfected quadratic may therefore be solved as follows:—

1. *Reduce and arrange it until all the terms involving x are on the first side, and the coefficient of x^2 is unity.*

2. *COMPLETE THE SQUARE by adding to each side the SQUARE OF HALF the coefficient of x .*

3. *Take the square root of each side, put a double sign to the second side, and transpose the term of the first side not involving x .*

Ex. 1. Solve the equation $3x^2 + 18x + 4 = 52$.

We have $3x^2 + 18x = 52 - 4 = 48$; or, dividing each side by 3, $x^2 + 6x = 16$.

Here, the coefficient of x is 6, the half of which is 3. Adding then the square of 3 to each side, to *complete the square*, we have—

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Taking the square root of each side, we have—

$$x + 3 = \pm 5.$$

$$\therefore x = \pm 5 - 3 = 2 \text{ or } -8.$$

Ex. 2. Given $\frac{x+3}{x+4} - \frac{x+1}{x+2} = \frac{4x+9}{2x+7} - \frac{12x+17}{6x+16}$,

find x .

$$\text{We have } \left(1 - \frac{1}{x+4}\right) - \left(1 - \frac{1}{x+2}\right) = \left(2 - \frac{5}{2x+7}\right)$$

$$- \left(2 - \frac{15}{6x+16}\right);$$

$$\text{or, simplifying, } \frac{1}{x+2} - \frac{1}{x+4} = \frac{15}{6x+16} - \frac{5}{2x+7};$$

$$\text{or, } \frac{(x+4) - (x+2)}{(x+2)(x+4)} = \frac{15(2x+7) - 5(6x+16)}{(6x+16)(2x+7)};$$

$$\text{or, simplifying, } \frac{2}{x^2 + 6x + 8} = \frac{25}{12x^2 + 74x + 112};$$

or, clearing of fractions—

$$24x^2 + 148x + 224 = 25x^2 + 150x + 200;$$

or, transposing and reducing, $-x^2 - 2x = -24$;

or, changing the sign of each side, then—

$$x^2 + 2x = 24;$$

or, completing the square, $x^2 + 2x + 1^2 = 24 + 1 = 25$.

Taking the square root of each side, we have—

$$x + 1 = \pm 5$$

$$\therefore x = \pm 5 - 1 = 4 \text{ or } -6.$$

Ex. 3. Solve the equation $x^2 + 6x + 25 = 0$.

We have $x^2 + 6x = -25$;

or, completing the square—

$$x^2 + 6x + 3^2 = -25 + 9 = -16;$$

or, extracting the square root of each side—

$$x + 3 = \pm \sqrt{-16}$$

$$\therefore x = -3 \pm \sqrt{-16}.$$

As the quantity $\sqrt{-16}$ has no exact or approximate value, the given equation has no real roots. The roots are therefore said to be imaginary.

73. *Solution by breaking into factors.*

We have seen (Art. 30) that it is often easy to find by inspection the factors of quadratic expressions. We may make use of this knowledge to solve quadratic equations.

Ex. 1. Solve the equation $x^2 + 5x = 66$.

Transposing all to the first side, we have—

$$x^2 + 5x - 66 = 0.$$

And, resolving the first side into its elementary factors, we get—

$$(x - 6)(x + 11) = 0.$$

Now, if either of these factors is put equal to 0, the equation is satisfied.

Hence, we have, $x - 6 = 0$, and $x + 11 = 0$;

or, $x = 6$, and $x = -11$.

$\therefore 6$ and -11 are the roots required.

Ex. 2. Given $x^2 - (a + b)x + ab = 0$, find x .

We have $(x - a)(x - b) = 0$.

Now this equation is satisfied by making either of the factors = 0.

$$\begin{aligned} \text{Hence, } x - a &= 0, \text{ or } x = a; \} \\ \text{and, } x - b &= 0, \text{ or } x = b; \} \end{aligned}$$

$\therefore a, b$ are the roots required.

Ex. XX.

1. $4x^2 - 7 = 29.$ 2. $5x^2 + 6 = 86.$

3. $\frac{3x^2 - 7}{2} = 34.$ 4. $\frac{x}{2} + \frac{3}{x} = 6.$

5. $3x - \frac{12}{x} = 0.$ 6. $2(a^2 + b^2) - x^2 = (a - b)^2.$

7. $* x\sqrt{24 + x^2} = 4 + x^2.$

8. $* \sqrt{x - a} = \sqrt{x - \sqrt{b^2 + x^2}}.$

9. $\frac{ab}{x} + \sqrt{\frac{a^2b^2}{x^2} - b^2} = x.$

10. $\frac{1}{ax - \sqrt{a^2x^2 - 1}} - \frac{1}{ax + \sqrt{a^2x^2 - 1}} = ax.$

11. $\frac{\sqrt[3]{a^2 + x^2} + \sqrt[3]{a^2 - x^2}}{\sqrt[3]{a^2 + x^2} - \sqrt[3]{a^2 - x^2}} = b.$

12. $\frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = a.$

13. $x^2 - 5x + 6 = 0.$

14. $x^2 - x = 72.$

15. $3x^2 - x = 2.$

16. $4x^2 - 9x = 28.$

17. $6x^2 + x - 35 = 0.$

18. $x^2 + 6\frac{1}{2}x + 8 = 0.$

19. $ax^2 + bx + c = 0.$

20. $3x - \frac{14}{2x - 3} = 4x - 7.$ 21. $x - \frac{42}{x} = 1.$

22. $2x + 1 - \frac{26}{2x + 1} = 11.$

* See Remark, page 113.

$$23. (a + bx)(cx + d) = (a + b)(c + dx).$$

$$24. (a - b)(x^2 - b^2) = (a + b)(x - b)^2.$$

$$25. (a + b)(x - a)(x - b) = abx.$$

$$26. \frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}.$$

$$27. ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}.$$

$$28. a^2x^2 - (a - b)^2x + a^3 = b^2x^2 + (a^2 + ab + b^2)x + b^3.$$

$$29. \frac{23}{x + 2} - \frac{2}{x} = 3\frac{1}{2}.$$

$$30. \frac{x + 1}{2x + 3} + \frac{7}{13} = \frac{2x + 5}{4x -}$$

$$31. \frac{2x + 9}{x} + \frac{x}{2x + 9} = 5\frac{1}{2}.$$

$$32. \frac{x + 2}{x + 3} + \frac{x - 10}{x + 8} = 0.$$

$$33. \frac{3x - 7}{4x - 2} + \frac{5x + 3}{7x + 4} = \frac{5}{9}.$$

$$34. \frac{6x + 5}{2x - 7} + \frac{4x - 1}{x - 2} = \frac{7x + 1}{x - 3}.$$

$$35. \frac{2x + 1}{x + 2} - \frac{x + 2}{4x + 4} = \frac{7x + 8}{4x + 13}.$$

$$36. \frac{3x}{x - 2} - \frac{2x + 9}{x + 2} = \frac{2x + 24}{2x - 1}.$$

$$37. \frac{3x^2 - 2x + 7}{6x^2 - 4x + 11} = \frac{x^2 + 7x - 3}{2x^2 + 14x - 9}$$

$$38. \frac{3x^2 + 10}{x} - \frac{x^2 + 2x + 3}{x + 2} = \frac{2x^2 + 2x + 10}{x + 1}.$$

$$39. * \sqrt{5x + 6} + \sqrt{x + 10} = 4.$$

$$40. \sqrt{3x - 4} - \sqrt{2x - 4} = \sqrt{x}.$$

$$41. * \sqrt{a^2 + b^2} + x + \sqrt{a^2 + b^2} - x = \sqrt{\frac{2ax}{b}}.$$

* See Remark, page 113.

✓ EQUATIONS WHICH MAY BE SOLVED LIKE QUADRATICS. 111

$$42. * \sqrt{2a+x} + \sqrt{4a-x} = 2\sqrt{x-a}.$$

$$43. * \sqrt{ax+b} + \sqrt{bx+a} = \sqrt{(a-b)x+a+b+2\sqrt{ab}}.$$

$$44. * \sqrt{x+\frac{b}{a}} + \sqrt{x+\frac{a}{b}} = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{b}}\right)\sqrt{a+b}.$$

$$45. \left(\frac{a}{b} - 1\right) \cdot \frac{x-c}{b} = \left(\frac{a^2}{b^2} + 1\right) \cdot \frac{a+b+c}{x} - 2.$$

$$46. (a-b)(x+b-c)(x+c-a) + a^2(x+b-c) + b^2(x+c-a) + c^2(x+a-b) = 0.$$

47. Show that, if x be real, the value of $x + \frac{1}{x}$ cannot lie between 2 and -2.

48. If, in the equation $x - \frac{x^2}{20} = a$, the quantity x be real, show that a cannot be greater than 5.

Equations which may be Solved like Quadratics.

74. Certain equations of a degree higher than the second may be solved like quadratics. It will be seen that, although it is impossible to lay down definite rules for the treatment of every such equation, the object to be attained is either (1.) To throw the equation into the form—

$$X^2 + pX = q,$$

when X is an expression containing the unknown quantity, and solving when possible this equation for X ; or (2.), To strike out from each side a factor containing the unknown quantity, thus reducing the dimensions of the equation, and obtaining a value or values of the unknown quantity by equating this factor to zero.

Ex. 1. Solve the equation $x^4 + 144 = 25x^2$.

We have, transposing, $x^4 - 25x^2 = -144$;

$$\text{or, } (x^2)^2 - 25(x^2) = -144;$$

* See Remark, page 113.

or, completing the square—

$$(x^2)^2 - 25(x^2) + \left(\frac{25}{2}\right)^2 = \frac{625}{4} - 144 = \frac{1}{4}.$$

$$\therefore x^2 - \frac{25}{2} = \pm \frac{1}{2}.$$

$$\therefore x^2 = \frac{25}{2} \pm \frac{1}{2} = 16 \text{ or } 9.$$

$$\text{Hence, } x = \pm 4 \text{ or } \pm 3.$$

The given equation has therefore *four* roots, being as many roots as the *degree* of the equation.

Ex. 2. Solve $x^6 + 3x^3 = 88$.

We have, $(x^3)^2 + 3(x^3) = 88$; or, transposing,

$$(x^3)^2 + 3(x^3) - 88 = 0;$$

$$\text{or, } (x^3 - 8)(x^3 + 11) = 0.$$

Hence, the given equation is satisfied by—

$$x^3 - 8 = 0, \text{ and also by } x^3 + 11 = 0.$$

We have then $x^3 - 2^3 = 0$, and $x^3 + (\sqrt[3]{11})^3 = 0$;

or, breaking into factors, we have—

$$(x - 2)(x^2 + x \cdot 2 + 2^2) = 0, \text{ and}$$

$$(x + \sqrt[3]{11}) \left\{ x^2 - x \cdot \sqrt[3]{11} + (\sqrt[3]{11})^2 \right\} = 0.$$

From the *first* of these equations, we get—

$$x - 2 = 0 \text{ or } x = 2,$$

and $x^2 + 2x + 4 = 0$, from which two other roots may be found.

And from the *second* equation, we get—

$$x + \sqrt[3]{11} = 0, \text{ or } x = -\sqrt[3]{11},$$

and $x^2 - x\sqrt[3]{11} + \sqrt[3]{121} = 0$, which gives two other roots.

We have therefore shown how to obtain the six roots of the given equation.

Ex. 3. Solve—

$$23x^3 - 75x - 6x\sqrt{4x^2 - 9x + 9} + 40 = 0.$$

Changing the sign of each side, and transposing, we have—

$$6x\sqrt{4x^2 - 9x + 9} - 9x + 9 = 23x^3 - 75x + 40;$$

adding to each side $13x^3 - 9x + 9$, then—

$$13x^3 - 9x + 9 + 6x\sqrt{4x^2 - 9x + 9} = 36x^3 - 84x + 49;$$

$$\text{or, } (4x^2 - 9x + 9) + 2(3x)\sqrt{4x^2 - 9x + 9} + (3x)^2 \\ = (6x)^2 - 2(6x) \cdot 7 + 7^2;$$

$$\text{or, } (\sqrt{4x^2 - 9x + 9} + 3x)^2 = (6x - 7)^2.$$

$$\therefore \sqrt{4x^2 - 9x + 9} + 3x = \pm (6x - 7), \dots\dots\dots(1.);$$

or, taking the upper sign—

$$\sqrt{4x^2 - 9x + 9} = (6x - 7) - 3x = 3 - 7.$$

Hence, squaring each side—

$$4x^2 - 9x + 9 = 9x^2 - 42x + 49;$$

$$\text{or, } 5x^2 - 33x + 40 = 0;$$

$$\text{or, } (x - 5)(5x - 8) = 0.$$

$$\therefore x = 5 \text{ or } \frac{8}{5}.$$

Again, taking the lower sign of (1.), we have—

$$\sqrt{4x^2 - 9x + 9} = -(6x - 7) - 3x = -9x + 7 \dots\dots\dots(2.);$$

or, squaring—

$$4x^2 - 9x + 9 = 81x^2 - 126x + 49;$$

$$\text{from which } 77x^2 - 117x + 40 = 0;$$

$$\text{or, } (x - 1)(77x - 40) = 0.$$

$$\therefore x = 1 \text{ or } \frac{40}{77}.$$

Hence, the roots of the given equation are $5, \frac{8}{5}, 1, \frac{40}{77}$.

REMARK.—If we proceed to verify these values of x , we shall find that the last two values—viz., 1 and $\frac{40}{77}$ —will not satisfy the given equation unless we obtain the value of $\sqrt{4x^2 - 9x + 9}$ by means of the equation from which these last roots were found.

Thus from (2.) we find, on putting 1 and $\frac{40}{77}$ successively for x —

$$\sqrt{4x^2 - 9x + 9} = -9(1) + 7 = -2,$$

$$\text{and } \sqrt{4x^2 - 9x + 9} = -9\left(\frac{40}{77}\right) + 7 = 2\frac{3}{77};$$

and on substituting either of these values of $\sqrt{4x^2 - 9x + 9}$ along with the corresponding value of x , the given equation is satisfied.

Ex. 4. Solve—

$$\begin{aligned} & (x + b + c)(x + b - c)(b + c - x) \\ &= (a + b + c)(a + b - c)(b + c - a). \end{aligned}$$

By inspection we see that a is one of the roots. We shall therefore so arrange the equation as to be able to strike out $x - a$ as a factor of each side (Art. 5).

We have—

$$\frac{(x + b + c)(x + b - c)}{(a + b + c)(a + b - c)} = \frac{b + c - a}{b + c - x};$$

$$\text{or, } \frac{(x + b)^2 - c^2}{(a + b)^2 - c^2} = \frac{b + c - a}{b + c - x};$$

or, taking the *difference* of numerator and denominator—

$$\frac{(x + b)^2 - (a + b)^2}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x};$$

$$\text{or, } \frac{(x - a)(x + 2b - a)}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x}.$$

Dividing each side by $x - a$, we have—

$$\frac{x + 2b - a}{(a + b)^2 - c^2} = \frac{1}{b + c - x}, \text{ and } x - a = 0, \text{ or } x = a;$$

Hence also $(x + 2b - a)(b + c - x) = (a + b)^2 - c^2$; an ordinary quadratic from which two other roots may be determined.

Ex. 5. * Solve the equation—

$$a^2 b^2 x^{\frac{1}{2}} - 4 a^{\frac{3}{2}} b^{\frac{3}{2}} x^{\frac{p+q}{2pq}} = (a - b)^2 x^{\frac{1}{2}}.$$

Dividing each side by $x^{\frac{1}{2}}$, we have—

$$a^2 b^2 x^{\frac{p-q}{2pq}} - 4 a^{\frac{3}{2}} b^{\frac{3}{2}} x^{\frac{p-q}{2pq}} = (a - b)^2;$$

$$\text{or, } \left(abx^{\frac{p-q}{2pq}} \right)^2 - 4 a^{\frac{3}{2}} b^{\frac{3}{2}} \left(abx^{\frac{p-q}{2pq}} \right) = (a - b)^2;$$

or, completing the square—

$$\begin{aligned} \left(abx^{\frac{p-q}{2pq}} \right)^2 - 4 a^{\frac{3}{2}} b^{\frac{3}{2}} \left(abx^{\frac{p-q}{2pq}} \right) + \left(2 a^{\frac{1}{2}} b^{\frac{1}{2}} \right)^2 \\ = (a - b)^2 + 4 ab = (a + b)^2. \end{aligned}$$

* We shall assume in this example that the laws of multiplication and division proved in Arts. 24, 28, hold for fractional indices.

Hence, extracting the square root—

$$abx^{\frac{p-q}{2pq}} - 2a^{\frac{1}{2}}b^{\frac{1}{2}} = \pm (a + b).$$

$$\therefore abx^{\frac{p-q}{2pq}} = \pm (a + b) + 2a^{\frac{1}{2}}b^{\frac{1}{2}} = (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 \text{ or } -(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2;$$

$$\begin{aligned} \therefore x^{\frac{p-q}{2pq}} &= \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}} \right)^2 \text{ or } - \left(\frac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}} \right)^2; \\ &= \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}} \right)^2 \text{ or } - \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}} \right)^2. \end{aligned}$$

Then, raising each side to the $(2pq)$ th power, we have—

$$x^{p-q} = \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}} \right)^{4pq} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}} \right)^{4pq}.$$

Hence, taking the $(p-q)$ th root, we get—

$$\begin{aligned} x &= \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}} \right)^{\frac{4pq}{p-q}} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}} \right)^{\frac{4pq}{p-q}}. \\ &= \left(\frac{1}{b^{\frac{1}{2}}} \pm \frac{1}{a^{\frac{1}{2}}} \right)^{\frac{4pq}{p-q}}. \end{aligned}$$

We have also, since the factor $x^{\frac{1}{2}}$ has been struck out—

$$x^{\frac{1}{2}} = 0, \text{ and } \therefore x = 0 \text{ as another solution.}$$

Ex. XXI.

1. $4x^4 - 11x^3 = 225.$
2. $5x^5 - 17x^3 = 184.$
3. $x^3 + 5x^{-3} = 25\frac{1}{2}.$
4. $(x^2 + 3x + 3)^2 + 2x^2 = 189 - 6x.$
5. $3x^{-4} - 17x^{-2} = 1450.$
6. $x^3 + \frac{6}{x^3} = 5.$
7. $\sqrt{x+12} + \sqrt{x+12} = 6.$
8. $\sqrt{x^2+9} + 20 = x^2 + 9.$
9. $\left(\frac{12}{x} - x\right)^2 + \frac{60}{x} = 6 + 5x.$

10. $(3x + 4) + \sqrt{3x + 4} - 12 = 0.$
11. $x^2 + 2x = \frac{2}{x + 4} - 1.$
12. $\frac{\sqrt[3]{a-x}}{x} - \frac{\sqrt[3]{a-x}}{a} = \frac{\sqrt[3]{x}}{a}.$
13. $9x + 24\sqrt{x} - \frac{4}{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + 9\right) = 65.$
14. $x = \sqrt{a} \cdot \sqrt{b} - x - \sqrt{b} \cdot \sqrt{x + a}.$
15. $\frac{1}{(a - \sqrt{a^2 - x^2})} + \frac{1}{(a + \sqrt{a^2 - x^2})} = \frac{a^2 + 2x^2}{x^4}.$
16. $\frac{x^2}{b} - 2\sqrt{\frac{a^2 - ax + x^2}{b}} = \frac{ax}{b} - 1.$
17. $x^{\frac{2+q}{2}} - \frac{1}{2} \cdot \frac{a^2 - b^2}{a^2 + b^2} (x^{\frac{1}{2}} + x^{\frac{1}{2}}) = 0.$
18. $\sqrt{\frac{a^2 + x^2}{x^2}} - \sqrt{\frac{x^2}{a^2 + x^2}} = 1.$
19. $(a^4 + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$
20. $\frac{x^2}{x^2 - 9} - \frac{975}{256} \cdot \frac{x^2 - 9}{x^2} + \frac{7}{8} = 0.$
21. $\frac{12}{x + 1} \left\{ \frac{3}{x + 1} + \frac{2}{x} \right\} + \frac{4}{x^2} = 9.$
22. $6x^2 - 5x - 8\sqrt{3x^2 + 5x - 4} = 12.$
23. $2x^2 + 5x - 2x\sqrt{2x^2 - 7x + 1} = 35.$
24. $20x^2 - 9ax - 8x\sqrt{5x^2 - 3ax + 2a^2} = 7a^2.$
25. $x^2(x - 2)^2 + 6x^2(x - 2) = 24x + 36 - 5x^2.$
26. $2x^2 + 20x - \sqrt{3x^2 - x + 5} = 105.$
27. $4x^2 + 12x - 2x\sqrt{4x^2 - 2x + 19} = 30.$
28. $x^4 + x^2 + 1 = a(x^2 + x + 1).$
29. $x^6 - 1 = 0.$

$$30. (x^2 - a)^2 + (x^2 - b)^2 + (x^2 - c)^2 \\ = 3(x^2 - a)(x^2 - b)(x^2 - c) + 9ax^4 - a(a + b + c)^2.$$

$$31. (x - b)(x^2 - c^2) = (a - b)(a^2 - c^2).$$

$$32. (x - a)(x - b)(x - c) = (m - a)(m - b)(m - c).$$

$$33. (x - a)(x - b)(x - c) = (a + b)(a + c)(b + c).$$

$$34. (n - 1)x(x^2 + ax + a^2) = a^3 - x^3.$$

$$35. (ax - b)^3 + (cx - d)^3 = (a + c)^3x^3 - (b + d)^3.$$

$$36. x^{2n} + x^{2n-1} + 1 = a(x^2 + x + 1).$$

Simultaneous Quadratic Equations.

75. The following worked examples are given as specimens of the methods to be employed, but it must be understood that practice alone will give the student complete mastery over equations of this class.

$$\begin{array}{l} \text{Ex. 1. Solve } x^2 + y^2 = 20 \dots\dots\dots(1.) \\ \quad \quad \quad x + y = 6 \dots\dots\dots(2.) \end{array} \left. \vphantom{\begin{array}{l} x^2 + y^2 = 20 \\ x + y = 6 \end{array}} \right\}$$

As we have given the *sum* of the unknown quantities, we shall work for the *difference*.

From (1.), multiplying each side by 2, we have—

$$\begin{array}{rcl} & 2x^2 & + 2y^2 = 40 \\ \text{and from (2.), squaring,} & x^2 + 2xy + y^2 & = 36 \\ \text{Then, subtracting,} & x^2 - 2xy + y^2 & = 4; \\ \text{and, taking the square root, we have} & x - y & = \pm 2(3.) \end{array}$$

(2.) + (3.), then $2x = 8$ or 4 , and $\therefore x = 4$ or 2 .

(2.) - (3.), then $2y = 4$ or 8 , and $\therefore y = 2$ or 4 .

NOTE.—Having found that $x = 4, y = 2$, we might have told by inspection that the values $x = 2, y = 4$, would also satisfy the given equations, for x and y are similarly involved in both equations.

$$\begin{array}{l} \text{Ex. 2. Solve } x^2 - y^2 = 5 \dots\dots\dots(1.) \\ \quad \quad \quad xy = 6 \dots\dots\dots(2.) \end{array} \left. \vphantom{\begin{array}{l} x^2 - y^2 = 5 \\ xy = 6 \end{array}} \right\}$$

As we have given here the *difference of the squares* of the unknown quantities, it will be convenient to work for the *sum of the squares*.

From (1.), squaring, $x^4 - 2x^2y^2 + y^4 = 25$,
 and from (2.), squaring, &c., $4x^2y^2 = 144$

Then, adding, $x^4 + 2x^2y^2 + y^4 = 169$
 and taking the square root, $x^2 + y^2 = \pm 13$ (3.).

(3.) + (1.), then, $2x^2 = 18$ or -8 , or $x^2 = 9$ or -4 ,
 and $\therefore x = \pm 3$ or $\pm 2\sqrt{-1}$.

(3.) - (1.), then, $2y^2 = 8$ or -18 , or $y^2 = 4$ or -9 ,
 and $\therefore y = \pm 2$ or $\pm 3\sqrt{-1}$.

NOTE.—The student will see that the pairs of values which satisfy the given equations are, $x = 3, y = 2$; $x = -3, y = -2$; $x = 2\sqrt{-1}, y = 3\sqrt{-1}$; $x = -2\sqrt{-1}, y = -3\sqrt{-1}$.

Ex. 3. Solve $x^2 + y = 11x$(1.) }
 $y^2 + x = 11y$(2.) }

Subtracting, then, $x^2 - y^2 - x + y = 11x - 11y$;
 or, $x^2 - y^2 = 12(x - y)$.

Now $(x - y)$ is a factor of each side, and hence, striking it out, we have—

$x + y = 12$(3.),
 and also $x - y = 0$(4.).

Equations (3.) and (4.) may not be used as simultaneous equations, but *each of them* may be used in turn with either of the given equations.

Thus, taking equations (3.) and (1.), we have—

(1.) - (3.), $x^2 - x = 11x - 12$,
 from which $x = 6 \pm 2\sqrt{6}$;

and hence from (3.), by substitution, we easily get—

$$y = 6 \mp 2\sqrt{6}.$$

Again, taking equations (4.) and (1.)—

we have, from (4.) $x = y$,

and $\therefore$ from (1.), $x^2 + x = 11x$ or $x^2 = 10x$,

from which $x = 10$ or 0 ;

and so, from (4.), $y = 10$ or 0 .

Hence, the pairs of values satisfying the given equations are—

$$x = 10, y = 10; x = 0, y = 0;$$

$$x = 6 + 2\sqrt{6}, y = 6 - 2\sqrt{6};$$

$$x = 6 - 2\sqrt{6}, y = 6 + 2\sqrt{6}.$$

NOTE.—It is worth while remarking that when *each of the terms* of the given equations contain at least one of the unknown quantities, the values $x = 0, y = 0$ will always satisfy.

$$\begin{aligned} \text{Ex. 4. Solve } 3x^2 - 2xy &= 55 \dots\dots\dots (1.) \\ x^2 - 5xy + 8y^2 &= 7 \dots\dots\dots (2.) \end{aligned}$$

Multiplying the equations together *crosswise*, we get—

$$55x^2 - 275xy + 440y^2 = 21x^2 - 14xy;$$

$$\text{or, transposing, } 34x^2 - 261xy + 440y^2 = 0;$$

$$\text{or, } (2x - 5y)(17x - 88y) = 0,$$

$$\text{from which } 2x = 5y, \text{ and } 17x = 88y.$$

Each of these equations taken *in turn* with either (1.) or (2.) will easily give the required values of x and y .

$$\begin{aligned} \text{Ex. 5. } x^4 + y^4 &= 337 \dots\dots\dots (1.), \\ x + y &= 7 \dots\dots\dots (2.) \end{aligned}$$

From (2.), raising each side to the *fourth power*, we have—

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401; \dots\dots\dots (3.)$$

$$(3.) - (1.), \text{ then } 4x^3y + 6x^2y^2 + 4xy^3 = 2064;$$

$$\text{or, } 2x^2y + 3x^2y^2 + 2xy^3 = 1032;$$

$$\text{or, arranging, } 2xy(x + y)^2 - x^2y^2 = 1032;$$

$$\text{but from (2.), } (x + y)^2 = 49,$$

$$\text{and hence, } 2xy(49) - x^2y^2 = 1032;$$

$$\text{or, } x^2y^2 - 98xy + 1032 = 0,$$

$$\text{from which } xy = 12 \text{ or } 86, \dots\dots\dots (4.)$$

From (2.) and (4.), $x - y$ may now be easily obtained, and hence also the required values of x and y .

Ex. XXII.

$$1. x + y = 5, xy = 6.$$

$$2. x - y = 2, xy = 15.$$

$$3. x^2 + y^2 = 25, xy = 12.$$

$$4. x^2 + y^2 = 20, x + y = 6.$$

5. $x^2 + y^2 = 29, x - y = 3.$
6. $x^2 - y^2 = 13, (x - y)^2 = 1.$
7. $x^2 - y^2 = 27, xy = 18.$
8. $x^2 - y^2 = 12, x + y = 6.$
9. $x^2 + y^2 = 53, x^2 - y^2 = -45.$
10. $x^2 + xy = 28, y^2 + xy = 21.$
11. $x^2 + xy + y^2 = 19, xy - x^2 = -3.$
12. $x + y = 13, \sqrt{x} + \sqrt{y} = 5.$
13. $x^2 + xy + y^2 = 84, x + \sqrt{xy} + y = 14.$
14. $x^3 + y^3 = 35, x^2y + xy^2 = 30.$
15. $\frac{1}{x} + \frac{1}{y} = a, \frac{1}{x^2} + \frac{1}{y^2} = b.$
16. $x + y = a(x - y), x^3 + y^3 = b^3.$
17. $x^4 - y^4 = a, x^2 - y^2 = b.$
18. $x + y = 5, x^3 + y^3 = 35.$
19. $x + y = 5, x^5 + y^5 = 275.$
20. $x^2 + y^2 = \frac{1}{8}(x + y), xy = 6.$
21. $x - y = 2, x^3 - y^3 = 98.$
22. $xy(x + y) = a, x^2y^2(x^2 + y^2) = b.$
23. $xy(x + y) = 30, x^2y^2(x^5 + y^5) = 9900.$
24. $4x^2 - 3xy = 18, 5y^2 - 2xy = 8.$
25. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 35, x^{\frac{1}{4}} + y^{\frac{1}{4}} = \frac{30}{x^{\frac{1}{4}}y^{\frac{1}{4}}}.$
26. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3x, x^{\frac{1}{4}} + y^{\frac{1}{4}} = x.$
27. $xy + 6 = \frac{216}{xy}, x + y + 4 = \frac{77}{x + y}.$
28. $(x + y)^2 + 2(x - y)^2 = 3(x + y)(x - y), x^2 + y^2 = 10.$
29. $x^2 + 10xy + y^2 = \frac{1}{5}(x^2 - y^2), x^2 + 5y^2 = x + 13y.$
30. $x^4 - 2x^2y^2 + y^4 = 1 + 4xy, x^2(x + 1) + y^2(y + 1) = xy.$
31. $3x + y - 9 = \frac{9x^2 - y^2 - 117}{3x - y - 6} = \frac{8x^2 - y^2 + 1}{4x - y + 1}.$

$$32. \frac{x+2}{y-2} + \frac{y+2}{x-2} = 5\frac{1}{2}, \frac{x+2}{y+2} + \frac{y-2}{x-2} = 1\frac{1}{2}.$$

$$33. \frac{\sqrt{x^2-y^2} + (x-y)}{\sqrt{x^2-y^2} - (x-y)} = 3 + 2\sqrt{2}, \frac{1}{y} - \frac{1}{x} = \frac{2}{xy}.$$

$$34. y - 2\sqrt{5x^2 + y + 3} = 32 - 5x^2,$$

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) = 14\frac{1}{2}.$$

$$35. \frac{x}{y} - \frac{y}{x} - 3 = -\frac{1}{2} \cdot \frac{xy}{x^2 - y^2}, 3x + y = 7.$$

$$36. (x+y)xy = c(bx+ay),$$

$$xy(bx+ay) = x^2y^2 + abc(x+y-c).$$

$$37. y^4 = x^2(ay-bx), x^3 = ax-bx.$$

$$38. \frac{\sqrt{y^2+1}+1}{y} = \frac{\sqrt{x+9}+3}{\sqrt{x}}, x(y+1)^2 = 36(y^2+16).$$

$$39. x = \frac{a}{y+z}, y = \frac{b}{x+z}, z = \frac{c}{x+y}.$$

$$40. x+y+z=6, xy+yz+zx=11, xyz=6.$$

$$41. x^2-yz=1, y^2-xz=0, z^2-xy=0.$$

$$42. xyz = a^2(x+y) = b^2(y+z) = c^2(x+z).$$

$$43. \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}.$$

$$44. x+y+z+u=4a+4b,$$

$$xy+yz+zx+yu+zu=6a^2+12ab+6b^2,$$

$$xyz+xyu+xzu+yzu=4a^3+12a^2b+12ab^2+4b^3,$$

$$xyzu=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

$$45. x^2y^2 + xy^2z + x^2yz = a,$$

$$y^2z^2 + xy^2z + xyz^2 = b,$$

$$x^2z^2 + x^2yz + xyz^2 = c.$$

$$46. (x+y)^3 + z^3 = 1125,$$

$$x+y+z=15,$$

$$xy=24.$$

$$47. \text{ If } ax^2 = by^2 = cz^2, \text{ and } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a, \text{ show that}$$

$$ax^3 + by^3 + cz^3 = (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 a^2.$$

48. Given $R = 1 + r$, $P = \frac{A}{r} (1 - R^{-n})$, $M = PR^n$,
show that $R = A \left(\frac{1}{P} - \frac{1}{M} \right) + 1$.

CHAPTER VIII.

Problems Producing Quadratic Equations.

76. We shall now discuss one or two problems whose solutions depend upon quadratic equations.

Ex. 1. A person raised his goods a certain rate per cent., and found that to bring them back to the original price he must lower them $3\frac{1}{2}$ less per cent. than he had raised them. Find the original rise per cent.

Let x = the original rise per cent.,

then $\frac{x}{100 + x} \cdot 100$ = the *fall per cent.* to bring them to the original price.

Hence, by problem—

$$x - \frac{100x}{100 + x} = 3\frac{1}{2}, \text{ which solved, gives}$$

$$x = 20 \text{ or } -16\frac{1}{2}.$$

The value $x = 20$ is alone applicable to the problem. Remembering, however, the algebraical meaning of the *negative* sign, it is easy to see that the second value, $x = -16\frac{1}{2}$, gives us the solution of the following problem:—

A person *lowered* his goods a certain rate per cent., and found that to bring them back to the original price he must *raise* them $3\frac{1}{2}$ more per cent. than he had *lowered* them. Find the original fall per cent.

The above solution tells us that the fall required is $16\frac{1}{2}$ per cent.

Had we worked the latter problem first, we should have obtained $x = 16\frac{1}{2}$ or -20 , the value $x = -20$ indicating the solution of the former problem.

Ex. 2. Find a number such that when multiplied by its deficiency from 100 the product is 196.

Let x = the number,
then $100 - x$ = its deficiency from 100.

Hence, by problem—

$$x(100 - x) = 196, \text{ or } x^2 - 100x + 196 = 0;$$

from which, $x = 2$ or 98 .

Both these values will be found to be consistent with the conditions of the problem.

Ex. 3. The number of men required to build a house is such that, when four times the number is subtracted from three times the square of the number, the result is 160. Find the number of men.

Let x = the number of men,
then, by problem—

$$3x^2 - 4x = 160, \text{ from which} \\ x = 8 \text{ or } -6\frac{2}{3}.$$

The value $x = 8$ is alone applicable to the problem as it stands. If, however, we may conceive of a *fractional* number of men—and this we may easily do here by supposing a *boy's* work to be equal to $\frac{2}{3}$ of a man's—we find that the second result gives us the solution of the following problem:—

The number of men required to build a house is such that when four times the number is *added* to three times the square of the number, the result is 160. Find the number.

The answer, as above indicated, is 6 men and 1 boy, where a boy is worth $\frac{2}{3}$ of a man.

The student will find, however, that in some cases there is no *obvious* interpretation to the second result, owing occasionally to the fact that certain *terms* are used in the problem to which the results will not apply, and indeed that the algebraical expression of the conditions of the problem is more general than the language of the problem itself.

Ex. XXIII.

1. Find a number whose square is equal to the product of two other numbers, one of which is less by 6 than the required number, and the other greater by 9 than twice that number.

2. When the numerator and denominator of a certain fraction are each increased by unity the fraction is increased by $\frac{1}{15}$, and when they are each diminished by unity the fraction is diminished by $\frac{1}{15}$. Find the fraction.

3. The mean proportional between the excess of a certain number above 21, and its defect from 37, is 28. Find the number.

4. A number of articles, which were bought for £4, cost each 3 shillings more than half as many shillings as there were articles. Find the number of articles.

5. There is a square court-yard, such that if its length be increased by 10 feet, and its breadth diminished by 20 feet, its area would be 5,104 feet. Required the side of the square.

6. If the number of shillings given for an article be added to the number of articles which can be bought at the same price for 18 shillings, the result is 11. Find the price.

7. Two travellers set out to meet each other from two places 180 miles distant; the first goes 3 miles an hour, and the second goes 1 mile more per hour than one-fourth of the number of hours before they meet. When will they meet?

8. A farmer bought a number of calves, sheep, and pigs, the number of calves being equal to that of the sheep and pigs together. For the calves he gave 64s. a head, and for the sheep twice as many shillings as there were sheep. He paid £153. 12s. for the calves and sheep together, and £36. 12s. for the pigs—a pig costing as much as a sheep and calf together. Find the cost of the sheep per head.

9. There are two squares, and an oblong whose sides are equal to those of the squares, and it is noted that three times the area of the first square exceeds four times the area of the oblong by 3 square feet, while twice the area of the square, together with three times the area of the rectangle, is 36 square feet. Required the sides of the squares.

10. The sum of two quantities is equal to 6 times the square of their product, and the sum of their cubes is equal to 36 times the product of their fifth powers. Find the quantities.

11. The solid content of a rectangular parallelopiped is 60 cubic feet, and the total area of the sides is 94 square feet, while the sum of the edges is 48 feet. Required the dimensions.

12. The products of the number of units of length in the sides of a polygon of n sides, when taken $n - 1$ together are respectively $a_1, a_2, a_3, \&c., a_n$. Required the lengths of the sides.

13. A, B, and C can together do a piece of work in a day, and C's rate of work is the product of the rates of A and B. Moreover, C is one-fifth as good a workman as A and B together. Find the respective times required for A, B, C to do a piece of work.

14. The compound interest of a certain sum of money for 3 years is a , and the third year's interest is b . Find the principal and the rate per cent.

15. A owes B £ a due m months hence, and also £ b due n months hence. Find the equated time, reckoning interest at 5 per cent. per annum.

16. Find three quantities such that the sum of any two is equal to the reciprocal of the third.

17. Find three magnitudes, when the quotients arising from dividing the products of every two by the other are respectively a, b, c .

CHAPTER IX.

THE THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS.

77. *A quadratic equation cannot have more than two roots.*

It has been shown (Art. 30) that $f(x)$ is divisible by $x - a$, when it vanishes on putting a for x ; that is, when a is a root of the equation $f(x) = 0$.

Hence, if α, β are roots of the quadratic equation

$$x^2 + px + q = 0,$$

$x - \alpha$ and $x - \beta$ are each of them factors of the expression $x^2 + px + q$, whatever be the value of x .

Moreover, $x - \alpha$ and $x - \beta$ are the only linear factors of $x^2 + px + q$, since their product gives an expression of two dimensions whose first term is x^2 , the same as that of the expression $x^2 + px + q$. We therefore have identically

$$x^2 + px + q = (x - \alpha)(x - \beta),$$

Hence no value of x can make $x^2 + px + q$ vanish unless

it make one of the factors $x - a$, $x - \beta$ vanish; that is, $x^2 + px + q$ cannot vanish for any other values of x than a and β .

It follows, therefore, that a, β are the only roots of the equation

$$x^2 + px + q = 0.$$

78. If a, β are the roots of the equation $x^2 + px + q = 0$, then $a + \beta = -p$, and $a\beta = q$.

For (Art. 77) we have *identically*,

$$\begin{aligned} x^2 + px + q &= (x - a)(x - \beta), \text{ or,} \\ &= x^2 - (a + \beta)x + a\beta. \end{aligned}$$

Here we have the coefficient p on the first side represented by $-(a + \beta)$ on the second side, and the constant term q represented by $a\beta$.

We then have

$$\begin{aligned} -(a + \beta) &= p, \text{ or } a + \beta = -p, \\ \text{and } a\beta &= q. \quad Q.E.D. \end{aligned}$$

COR. 1. It easily follows that

$$\begin{aligned} (1.) \quad a^2 + \beta^2 &= p^2 - 2q. \\ (2.) \quad a - \beta &= \sqrt{p^2 - 4q}. \\ (3.) \quad \frac{1}{a} + \frac{1}{\beta} &= \frac{a + \beta}{a\beta} = -\frac{p}{q} \end{aligned}$$

COR. 2. If the given equation be of the form $ax^2 + bx + c = 0$, we may write it thus:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

and we have

$$\begin{aligned} (1.) \quad a + \beta &= -\frac{b}{a} \\ (2.) \quad a\beta &= \frac{c}{a} \\ (3.) \quad a^2 + \beta^2 &= \frac{b^2 - 2ac}{a^2} \end{aligned}$$

$$(4.) \quad \alpha - \beta = \frac{1}{a} \sqrt{b^2 - 4ac}.$$

$$(5.) \quad \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}.$$

79. *The quadratic expression $ax^2 + bx + c$ is a perfect square when $b^2 = 4ac$.*

When $ax^2 + bx + c$ is a perfect square, its *linear* factors must be identical; and hence the condition is that the equation

$$ax^2 + bx + c = 0$$

must have *equal roots*.

Solving this equation we get

$$x = \frac{1}{2a} (-b \pm \sqrt{b^2 - 4ac}).$$

Now these two values of x can be equal only when

$$b^2 - 4ac = 0, \text{ or when } b^2 = 4ac.$$

Hence the condition that $ax^2 + bx + c$ shall be a perfect square is that $b^2 = 4ac$.

COR. The roots of the equation $ax^2 + bx + c = 0$ are *rational* when $b^2 - 4ac$ is a *perfect square*, and are *impossible* when $b^2 < 4ac$.

80. *To investigate the relations between the coefficients A, B, C, a, b, c, in order that the expression*

$$Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy$$

shall be a perfect square.

We must have *identically*

$$\begin{aligned} & Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy \\ &= (\sqrt{A} \cdot x + \sqrt{B} \cdot y + \sqrt{C} \cdot z)^2 \\ &= Ax^2 + By^2 + Cz^2 + 2\sqrt{BC} \cdot yz + 2\sqrt{AC} \cdot xz \\ &\quad + 2\sqrt{AB} \cdot xy. \end{aligned}$$

Equating the coefficients of identical terms, we have

$$a = 2\sqrt{BC}, \quad b = 2\sqrt{AC}, \quad c = 2\sqrt{AB},$$

the relations required.

COR. Solving for A, B, C, we may put these relations in the following form :

$$A = \frac{bc}{2a}, \quad B = \frac{ac}{2b}, \quad C = \frac{ab}{2c}.$$

81. *The values of x and y derived from the equations*

$$ax + by = \frac{a^2}{x} + \frac{b^2}{y} = c^2$$

are rational, when $a^2 + b^2 = c^2$.

For, solving for x, we get—

$$ac^2x^2 + (b^4 - c^4 - a^4)x + a^3c^2 = 0.$$

Hence (Art. 79, Cor.) the values of x are rational when $(b^4 - c^4 - a^4)^2 - 4ac^2 \cdot a^3c^2$ is a perfect square.

The values of x are therefore rational when

$$(b^4 - c^4 - a^4)^2 = 4a^4c^4, \text{ or}$$

$$b^4 - c^4 - a^4 = \pm 2a^2c^2, \text{ or}$$

$$(a^2 \pm c^2)^2 = b^4, \text{ or when } a^2 \pm c^2 = \pm b^2.$$

And we may similarly show that the values of y are rational when

$$b^2 \pm c^2 = \pm a^2.$$

Now, on examination it will be found that the only condition common to these two sets is that

$$a^2 + b^2 = c^2.$$

Hence the values of x and y are both rational when

$$a^2 + b^2 = c^2. \quad Q.E.D.$$

82. *To investigate the condition that the expression*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

is resolvable into elementary factors.

If the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be solved either for x or y, the condition that either can be expressed in terms of the other in a rational form will (Art. 1) be the condition that the given expression can be broken up into elementary rational factors.

Solving for x , we find x equal to

$$\frac{-By + D \pm \sqrt{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)}}{2A}.$$

Now, in order that this may be rational, we must have

$$(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)$$

a perfect square.

Hence (Art. 79) we must have

$$\{2(BD - 2AE)\}^2 = 4(B^2 - 4AC)(D^2 - 4AF)$$

$$\text{or} \quad (BD - 2AE)^2 = (B^2 - 4AC)(D^2 - 4AF) \dots (1).$$

This may be transformed into either of the following forms:

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0 \dots \dots (2).$$

$$(BE - 2CD)^2 = (B^2 - 4AC)(E^2 - 4CF) \dots \dots (3).$$

$$(DE - 2BF)^2 = (D^2 - 4AF)(E^2 - 4CF) \dots \dots (4).$$

Ex. XXIV.

1. Find the difference of the squares of the roots of the equation $3x^2 - 12x - 43 = 0$.

2. Construct an equation whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$.

3. If α, β are the roots of the equation $x^2 - px + q = 0$, construct the equation whose roots are $\alpha - \gamma, \beta - \gamma$.

4. Find the value of α when the equation

$$ax^2 - 36x + 81 = 0$$

has equal roots.

5. Find the relation between the quantities a, b, c, a', b', c' , so that the equations

$$ax^2 + bx + c = 0, \text{ and } a'x^2 + b'x + c' = 0$$

shall have a common root.

6. The equations $ax^2 + bx + c = 0$, and $2ax + b = 0$, have common roots when the former has two equal roots.

7. When is the equation

$$\sqrt{m^2x + a^2} + \sqrt{n^2x + b^2} + \sqrt{p^2x + c^2} = 0$$

reducible to a simple equation?

8. If α, β are the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $\alpha + \beta$, and $\alpha - \beta$.

9. If $x^2 + px + q$ and $x^2 + mx + n$ have a common factor, show that this common factor is $x + \frac{n - q}{m - p}$.

10. Hence show that the following relation holds, viz.,
 $(n - q)^2 + n(m - p)^2 = m(m - p)(n - q)$.

11. Show that the values of x and y derived from the equations $ax^3 + by^3 = a^3x + b^3y = c^3$ are rational when $a^3 + b^3 = c^3$.

12. Show that the expression $6x^2 + 17xy + 12y^2 + 4x + 7y - 10$ can be broken into elementary factors, and determine the factors.

13. Show that the equations

$$\begin{aligned}x^3 - 3x^2 + 5x - 36 &= 0 \\5x^3 + 7x^2 - 432 &= 0\end{aligned}$$

have a common root.

14. If α, β are the roots of the equation $x^3 + x + 1 = 0$, show that $\alpha^3 - \beta^3 = 0$.

15. The equation $ax^3 + bx + c = 0$, can be transformed into an equation of the form $Ax^3 + B = 0$, by making $x = x' - \frac{b}{3a}$.

16. If α, β are the roots of the equation $ax^3 + bx + c = 0$, then

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{b^2}{ca} - 2.$$

17. The sum of the factors of the expression
 $bx^3 - xy - 2y^2 - 11x + 19y - 35$
 is $5x - y + 2$.

18. The equation $ax^3 + bx^2 + cx + d = 0$, has its roots m times as large as the roots of the equation

$$am^3x^3 + bm^2x^2 + cmx + d = 0.$$

CHAPTER X.

THEORY OF INDICES.

Indices.

83. Part of the matter of this chapter has been to a certain extent anticipated.

I. *Positive integral indices.*

It is convenient to conceive of a^m , where m is a positive integer, as the result of *multiplying* unity by m factors each equal to a .

Thus $a^m = 1 (a . a . a \text{ to } m \text{ factors})$.

II. *Negative integral indices.*

Here, remembering that *negative* is the *reverse* of *positive*, we shall be quite consistent if we allow a^{-m} to represent the result of *dividing* unity by a , and the quotients successively by a for m operations.

Thus, $a^{-m} = 1 \div (a . a . a \text{ to } m \text{ factors}) = \frac{1}{a^m}$.

We get from this $a^m \times a^{-m} = 1$.

Now, by *assuming* that the formula $a^m \times a^n = a^{m+n}$, which we know holds for positive integral indices, to also hold when

$$n = -m, \text{ we have } a^m \times a^{-m} = a^{m-m} = a^0;$$

and we conclude that $a^0 = 1$.

We have no need, however, to *assume* the principle at all.

For, referring to our definitions of a^m and a^{-m} , it at once follows, since a^m represents the result of *multiplying* unity by a , and the successive products by a for m operations, and that a^{-m} represents the result of *dividing* unity by a , and the successive quotients by a for m operations, that a^0 represents unity *neither multiplied nor divided at all* by a ; that is, a^0 is consistently interpreted to mean unity.

Hence, whether m be a positive or negative integer, or zero, a^m admits of the same interpretation, provided we remember the algebraical meaning of *positive* and *negative*.

Again, the law that $a^m \times a^n = a^{m+n}$ holds good if either m or n , or both, be negative integers.

For, suppose $n = -p$, when p is a positive integer, we have

$$a^m \times a^n = a^m \times a^{-p} = a^m \times \frac{1}{a^p} = \frac{a^m}{a^p}.$$

$$\text{If } m > p, \text{ then } \frac{a^m}{a^p} = a^{m-p} = a^{m+n}.$$

$$\text{If } m < p, \text{ then } \frac{a^m}{a^p} = \frac{1}{a^{p-m}} = a^{-(p-m)} = a^{m-p} = a^{m+n}.$$

Hence $a^m \times a^n = a^{m+n}$, when n is negative.

And it may be similarly shown when both are negative.

Fractional Indices.

84. If m and n are any integers, it follows at once from the above that

$$(a^m)^n = a^{mn} = a^{m \times n}.$$

Hence, a^{mn} or $a^{m \times n}$ is the correct symbol for the n th power of a^m . And therefore a^{m+n} or $a^{\frac{m}{n}}$ will consistently represent the n th root of a^m . And this is true whether m and n be either one or both positive or negative.

(i.) We shall now show that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q} + \frac{r}{s}}$, where p, q, r, s are integers, and thus show that the law $a^m \times a^n = a^{m+n}$ holds when m and n are fractional.

$$\text{We have } a^{\frac{p}{q}} = \sqrt[q]{a^p} = \sqrt[q]{(a^p)^s} = \sqrt[q]{a^{ps}} \dots\dots\dots(1).$$

$$\text{and } a^{\frac{r}{s}} = \sqrt[s]{a^r} = \sqrt[s]{(a^r)^q} = \sqrt[s]{a^{qr}} \dots\dots\dots(2).$$

$$\begin{aligned} \therefore a^{\frac{p}{q}} \times a^{\frac{r}{s}} &= \sqrt[q]{a^{ps}} \times \sqrt[s]{a^{qr}} = \sqrt[q]{a^{ps+qr}} \\ &= a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q} + \frac{r}{s}}. \end{aligned}$$

(ii.) We shall now show that $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$, where p, q, r, s are integers, and thus that the law $(a^m)^n = a^{mn}$ holds when m and n are fractional.

We have $(a^{\frac{p}{q}})^{\frac{r}{s}} = \sqrt[s]{\sqrt[q]{a^{\frac{p}{q}}}}^r$,
or by what has just been proved,

$$\begin{aligned} &= \sqrt[q]{a^{\frac{pr}{q}}} \\ &= \sqrt[q]{\sqrt[s]{a^{\frac{pr}{qs}}}} = \sqrt[qs]{a^{\frac{pr}{qs}}} = a^{\frac{pr}{qs}}. \end{aligned}$$

Hence it has been established that the laws

(i.) $a^m a^n = a^{m+n}$, and

(ii.) $(a^m)^n = a^{mn}$,

are universally true, whether m, n be integral or fractional, positive or negative.

Ex. XXV.

Find the value of—

1. $(a^6)^{\frac{1}{2}}, a^{\frac{2}{3}}, (a^{-2})^{-\frac{1}{2}}, (a^{\frac{1}{2}})^{\frac{2}{3}}$.

2. $(a+x)^{\frac{1}{2}}(a^2+2ax+x^2)^{\frac{1}{2}}, (a^{\frac{1}{2}}-x^{\frac{1}{2}})^{\frac{1}{2}}(a^{\frac{1}{2}}+x^{\frac{1}{2}})^{\frac{1}{2}}$.

Multiply together—

3. $a^{\frac{1}{2}}-a^{\frac{1}{2}}x^{\frac{1}{2}}+x^{\frac{1}{2}}$ and $a^{\frac{1}{2}}+a^{\frac{1}{2}}x^{\frac{1}{2}}+x^{\frac{1}{2}}$.

4. $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ and $x^{-\frac{1}{2}}+y^{-\frac{1}{2}}$.

5. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ and $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

Divide—

6. a^2-b by $a-b^{\frac{1}{2}}$, $a^{\frac{2}{3}}-b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

7. $x^{\frac{2}{3}}-xy^{\frac{1}{3}}+x^{\frac{1}{3}}y-y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+x^{-\frac{1}{3}}y$.

8. $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

9. $a^{\frac{2}{3}}-a^{\frac{2}{3}}b^{-\frac{1}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{3}}-b^{-\frac{1}{3}}$.

Show that—

$$10. (x^m + x^n)^{\frac{1}{mn}} = x^{\frac{1}{m}} + \frac{1}{n} \left\{ x^{m-n} + x^{n-m} \right\}^{\frac{1}{mn}}.$$

$$11. \frac{a^2 - b^2}{a^2 + b^2} \left\{ x^{\frac{1}{p}} + x^{\frac{1}{q}} \right\} - 2 x^{\frac{p+q}{2pq}} \\ = \frac{a^2 - b^2}{a^2 + b^2} \left\{ x^{\frac{1}{2p}} - x^{\frac{1}{2q}} \right\} - \frac{4b^2}{a^2 + b^2} x^{\frac{p+q}{2pq}}.$$

$$12. (x^m - a^m) \div (x^{\frac{m}{2}} - a^{\frac{m}{2}}) \\ = (x^{\frac{m}{2}} + a^{\frac{m}{2}}) (x^{\frac{m}{4}} + a^{\frac{m}{4}}) (x^{\frac{m}{8}} + a^{\frac{m}{8}}) \dots (x^{\frac{m}{n}} + a^{\frac{m}{n}}).$$

Find the square roots of—

$$13. a + b + c + 2(a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}c^{\frac{1}{2}} + b^{\frac{1}{2}}c^{\frac{1}{2}}).$$

$$14. 4xy^{\frac{1}{2}} - 12x^{\frac{3}{2}}y + 17x^2y^{\frac{1}{2}} - 12x^{\frac{5}{2}}y^{\frac{1}{2}} + 4x^3.$$

$$15. a^2b^{-1} - 4ab^{-\frac{1}{2}} - 8a^{-1}b^{\frac{1}{2}} + 4a^{-2}b + 8.$$

Find the cube roots of—

$$16. x^4 + 9x^{\frac{1}{2}} + 6x^{\frac{1}{3}} - 99x^2 - 42x^{\frac{1}{2}} + 441x^{\frac{1}{3}} - 343.$$

$$17. x^2y^{-1} + 3x^4y^{-\frac{2}{3}} + 3x^2y^{-\frac{1}{3}} + 1.$$

$$18. ab(1 + 3a^{-\frac{1}{2}}b^{\frac{1}{2}} + 3a^{-\frac{1}{2}}b^{\frac{1}{2}} + a^{-1}b)(ab^{-1} - 3a^{\frac{1}{2}}b^{-\frac{1}{2}} \\ + 3a^{\frac{1}{2}}b^{-\frac{1}{2}} + 1).$$

CHAPTER XI.

SURDS—IMAGINARY QUANTITIES.

85. A surd quantity is one in which the root indicated cannot be denoted without the use of a fractional index.

Thus, the following quantities are surds :—

$$\sqrt{a}, \sqrt[4]{a^2 + x^2}, \sqrt[3]{a^2 + b^2 + c^2}, \sqrt{\frac{x+y}{a}}, \frac{\sqrt[4]{a+x}}{\sqrt[3]{x+y}}$$

Since, from what has been explained in the last chapter, these quantities may be written thus—

$$a^{\frac{1}{2}}, (a^2 + x^2)^{\frac{1}{2}}, (a^2 + b^2 + c^2)^{\frac{1}{2}}, \left(\frac{x+y}{a}\right)^{\frac{1}{2}}, \frac{(a+x)^{\frac{1}{2}}}{(x+y)^{\frac{1}{2}}},$$

it follows that surds may be dealt with exactly as we deal with their equivalent expressions with fractional indices.

It is evident that rational quantities may be put in the form of surds, and conversely, expressions which have the form of surds may sometimes be rational quantities.

$$\text{Thus, } a^2 = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6};$$

$$\text{and } \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3} = \sqrt[3]{(a+b)^3} = a+b.$$

86. A mixed quantity may be expressed as a surd.

$$\text{Thus, } 3\sqrt[3]{5} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{3^3 \times 5} = \sqrt[3]{135},$$

$$\text{and so, } x\sqrt[3]{y} = \sqrt[3]{x^3 y}. \quad \sqrt[3]{y} = \sqrt[3]{x^3 y}.$$

87. Conversely, a surd may be expressed as a mixed quantity, when the root of any factor can be obtained.

$$\begin{aligned} \text{Thus, } \sqrt{18a^2b^2} &= \sqrt{9a^2b^2 \times 2a} \\ &= \sqrt{9a^2b^2} \cdot \sqrt{2a} = 3ab\sqrt{2a}. \end{aligned}$$

$$\begin{aligned} \text{And } \sqrt[3]{(a^2 + b^2)^6 x^4 y^5} &= \sqrt[3]{(a^2 + b^2)^6 x^3 y^3 \times xy^2} \\ &= \sqrt[3]{(a^2 + b^2)^6 x^3 y^3} \cdot \sqrt[3]{xy^2} = (a^2 + b^2)^2 xy \sqrt[3]{xy^2}. \end{aligned}$$

88. Fractional surd expressions may be so expressed that the surd portion may be integral.

The process is called *rationalizing the denominator*, and is worth special notice.

$$\text{Ex. 1. } \sqrt[3]{\frac{3}{7}} = \sqrt[3]{\frac{3 \times 7}{7^3}} = \frac{\sqrt[3]{21}}{\sqrt[3]{7^3}} = \frac{\sqrt[3]{21}}{7}.$$

It is much easier to find approximately the value of $\sqrt[3]{21}$, and divide the result by 7, than to find the values of $\sqrt[3]{3}$ and $\sqrt[3]{7}$, and divide the former by the latter.

$$\text{Ex. 2. Reduce to its simplest form } \sqrt{\frac{xy}{b-c}}.$$

$$\sqrt{\frac{xy}{b-c}} = \sqrt{\frac{xy(b-c)}{(b-c)^2}} = \frac{\sqrt{xy(b-c)}}{b-c}.$$

Ex. 3. Find the arithmetical value of $\frac{4}{2 - \sqrt{3}}$.

The denominator is the *difference* of two quantities, one of which is a *quadratic* surd.

Now, we know that $(2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$, and hence we see that by multiplying numerator and denominator by the *sum* of the quantities in the denominator we can obtain the denominator in a *rational form*.

$$\begin{aligned}\text{Thus, } \frac{4}{2 - \sqrt{3}} &= \frac{4(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{4(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2} \\ &= \frac{4(2 + \sqrt{3})}{4 - 3} = \frac{4(2 + \sqrt{3})}{1} = 4(2 + \sqrt{3}) \\ &= 4(2 + 1.73205) = 4(3.73205) \\ &= 14.92820.\end{aligned}$$

$$\begin{aligned}\text{And so, } \frac{4}{4\sqrt{2} + 3\sqrt{3}} &= \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2} + 3\sqrt{3})(4\sqrt{2} - 3\sqrt{3})} \\ &= \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2})^2 - (3\sqrt{3})^2} = \frac{4(4\sqrt{2} - 3\sqrt{3})}{32 - 27} = \frac{4}{5}(4\sqrt{2} - 3\sqrt{3}).\end{aligned}$$

We shall now give an example when the surds are not quadratic.

Ex 4. Rationalize the denominator of $\frac{a}{x^{\frac{1}{3}} - y^{\frac{1}{3}}}$.

Since $(x^{\frac{1}{3}})^3 - (y^{\frac{1}{3}})^3$ is (Art. 29) divisible by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$, it follows that the rationalizing factor is their quotient, which is easily found.

89. *Surds may be reduced to a common index.*

Ex. 1. Express $\sqrt[n]{a}$ and $\sqrt[m]{b}$ as surds having a common index.

Since $\sqrt[n]{a} = a^{\frac{1}{n}}$, and $\sqrt[m]{b} = b^{\frac{1}{m}}$, it follows that, by reducing the fractional indices to a common denominator, the given surds become respectively $a^{\frac{n}{mn}}$, $b^{\frac{m}{mn}}$, or $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

Ex. 2. Reduce $\sqrt[4]{a^3b}$ and $\sqrt[3]{x^5y^2}$ to a common index.

The least common denominator of the fractional indices of the given surds is 4×3 or 12. Hence we proceed as follows:—

$$\begin{aligned}\sqrt[4]{a^3b} &= (a^3b)^{\frac{1}{4}} = (a^3b)^{\frac{3}{12}} = \sqrt[12]{(a^3b)^3} = \sqrt[12]{a^9b^3}, \\ \sqrt[3]{x^5y^2} &= (x^5y^2)^{\frac{1}{3}} = (x^5y^2)^{\frac{4}{12}} = \sqrt[12]{(x^5y^2)^4} = \sqrt[12]{x^{20}y^8}.\end{aligned}$$

When the student has had a little practice, the first two steps of each of the operations may be omitted.

90. Addition and subtraction of similar surds.

DEF. Similar surds are those which have the same irrational factors.

Ex. 1. Find the sum of $\sqrt{12}$, $5\sqrt{27}$, $-2\sqrt{75}$.

We have—

$$\begin{aligned}&\sqrt{12} + 5\sqrt{27} - 2\sqrt{75} \\&= \sqrt{2^2 \times 3} + 5\sqrt{3^2 \times 3} - 2\sqrt{5^2 \times 3} \\&= 2\sqrt{3} + 5 \times 3\sqrt{3} - 2 \times 5\sqrt{3} \\&= 2\sqrt{3} + 15\sqrt{3} - 10\sqrt{3} \\&= (2 + 15 - 10)\sqrt{3} = 7\sqrt{3}.\end{aligned}$$

Ex. 2. Simplify—

$$\sqrt{\frac{a^2b + 2ab^2 + b^3}{a^2 - 2ab + b^2}} - \sqrt{\frac{a^2b - 2ab^2 + b^3}{a^2 + 2ab + b^2}}.$$

The given expression—

$$\begin{aligned}&= \sqrt{\frac{(a+b)^2b}{(a-b)^2}} - \sqrt{\frac{(a-b)^2b}{(a+b)^2}} \\&= \frac{a+b}{a-b} \sqrt{b} - \frac{a-b}{a+b} \sqrt{b} = \left(\frac{a+b}{a-b} - \frac{a-b}{a+b} \right) \sqrt{b} \\&= \frac{(a+b)^2 - (a-b)^2}{(a-b)(a+b)} \sqrt{b} = \frac{4ab}{a^2 - b^2} \sqrt{b}.\end{aligned}$$

91. Multiplication and division of surds.

The following examples will best illustrate these operations:—

Ex. 1. Multiply $a\sqrt{x^2yz}$ by $b\sqrt{xy^2u}$.

$$\begin{aligned}\text{We have, } a\sqrt{x^2yz} \times b\sqrt{xy^2u} &= ab\sqrt{x^2yz \times xy^2u} \\ &= ab\sqrt{x^3y^3uz} = abx^2y^2\sqrt{uz}.\end{aligned}$$

Ex. 2. Multiply $a\sqrt{b} + c\sqrt{d}$ by $a - \sqrt{bd}$.

Arranging as in the case of rational quantities, we have—

$$\begin{array}{r} a\sqrt{b} + c\sqrt{d} \\ a - \sqrt{bd} \\ \hline a^2\sqrt{b} + ac\sqrt{d} \\ - ab\sqrt{d} - cd\sqrt{b} \\ \hline (a^2 - cd)\sqrt{b} + a(c - b)\sqrt{d} \end{array}$$

Ex. 3. Divide $a\sqrt{b}$ by $b\sqrt{a}$.

$$\text{We have, } \frac{a\sqrt{b}}{b\sqrt{a}} = \frac{a\sqrt{b} \cdot \sqrt{a}}{b\sqrt{a} \cdot \sqrt{a}} = \frac{a\sqrt{ab}}{b \cdot a} = \frac{1}{b} \sqrt{ab}.$$

When the divisor is a compound quantity it will *generally* be the best to express the surds as quantities with fractional indices, and proceed as in ordinary division.

92. *The square root of a rational quantity cannot be partly rational and partly irrational.*

1. possible, let $\sqrt{a} = m + \sqrt{b}$;

then, squaring, $a = m^2 + 2m\sqrt{b} + b$;

or, $2m\sqrt{b} = a - (m^2 + b)$;

or, $\sqrt{b} = \frac{a - (m^2 + b)}{2m}$;

that is, an irrational quantity is equal to a rational quantity, which is absurd.

93. *To find the square root of a binomial, one of whose terms is a quadratical surd.*

Let $a + \sqrt{b}$ be the binomial.

Assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, ...(1);

then, squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$, ...(2).

Equating the rational and irrational parts (Art. 92), we have—

$$x + y = a \dots\dots\dots(3.),$$

$$\text{and } 2\sqrt{xy} = \sqrt{b} \text{ or } 4xy = b \dots\dots\dots(4.)$$

From (3) and (4) we easily find $x = \frac{1}{2}(a + \sqrt{a^2 - b})$,
and $y = \frac{1}{2}(a - \sqrt{a^2 - b})$.

Hence, from (1), the square root required is—

$$\sqrt{\frac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\frac{1}{2}(a - \sqrt{a^2 - b})}.$$

NOTE.—It is evident that, unless $(a^2 - b)$ is a perfect square, our result is more complicated than the original expression, and therefore the above method fails in that case.

Ex. 1. Find the square root of $14 + 6\sqrt{5}$.

$$\text{Let } \sqrt{14 + 6\sqrt{5}} = \sqrt{x} + \sqrt{y} \dots\dots\dots(1.)$$

Squaring, then, $14 + 6\sqrt{5} = x + y + 2\sqrt{xy}$.

Hence, equating the rational and irrational parts—

$$x + y = 14 \dots\dots\dots(2.),$$

$$2\sqrt{xy} = 6\sqrt{5} \text{ or } 4xy = 180 \dots\dots\dots(3.)$$

From (2) and (3) we easily find $x = 9, y = 5$.

Hence the square root required is $\sqrt{9} + \sqrt{5}$ or $3 + \sqrt{5}$.

Ex. 2. Find the square root of $39 + \sqrt{1496}$.

$$\text{Let } \sqrt{39 + \sqrt{1496}} = \sqrt{x} + \sqrt{y}.$$

Squaring, &c., we have, $x + y = 39$;
and $4xy = 1496$.

From these equations we easily find $x = 22, y = 17$.

Hence, the square root required is $\sqrt{22} + \sqrt{17}$.

94. *The square roots of quantities of this kind may often be found by inspection.*

Ex. 1. Find the square root of $19 + 8\sqrt{3}$.

We shall throw this expression into the form $a^2 + 2ab + b^2$, which we know is a perfect square.

Dividing the irrational term by 2, we have $4\sqrt{3}$. Now all we have to do is to break this up into two such factors that the sum of their squares shall be 19. The factors are evidently 4 and $\sqrt{3}$.

$$\begin{aligned}\text{Thus, we have } 19 + 8\sqrt{3} &= (4)^2 + 2(4)\sqrt{3} + (\sqrt{3})^4 \\ &= (4 + \sqrt{3})^2.\end{aligned}$$

The square root is therefore $4 + \sqrt{3}$.

Ex. 2. Find the square root of $29 + 12\sqrt{5}$.

$$\begin{aligned}\text{We have } 29 + 12\sqrt{5} &= (3)^2 + 2(3)2\sqrt{5} + (2\sqrt{5})^2 \\ &= (3 + 2\sqrt{5})^2.\end{aligned}$$

The square root is therefore $3 + 2\sqrt{5}$.

Imaginary Quantities.

95. An expression of the form $a + b\sqrt{-1}$ is called an **imaginary quantity**, a and b being real quantities. If we allow that $b\sqrt{-1}$ vanishes when $b = 0$, then the whole expression vanishes when $a = 0$, and $b = 0$.

Conversely, when $a + b\sqrt{-1} = 0$, we must have $a = 0$, $b = 0$.

For, transposing, $a = -b\sqrt{-1}$; or, squaring,

$$\begin{aligned}a^2 &= b^2(-1) = -b^2; \text{ or,} \\ a^2 + b^2 &= 0.\end{aligned}$$

And the only values of a and b which satisfy this equality are

$$a = 0, b = 0.$$

96. If $a + b\sqrt{-1} = c + d\sqrt{-1}$, then $a = c$, and $b = d$.

For, transposing,

$$(a - c) + (b - d)\sqrt{-1} = 0.$$

Hence, by the last Art., $a - c = 0$, or $a = c$,
and $b - d = 0$, or $b = d$.

97. DEF.—Two imaginary expressions are **conjugate** when they differ only in the sign of the coefficient of $\sqrt{-1}$.

Thus $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are conjugate expressions.

It follows, therefore, that

(i.) *The sum of two conjugate imaginary expressions is REAL.*

For $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$.

(ii.) *The difference of two conjugate imaginary expressions is IMAGINARY.*

The square root of the product of an imaginary expression by its conjugate is called its **modulus**.

Thus, since $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$,
we call $\sqrt{a^2 + b^2}$ the modulus of $a + b\sqrt{-1}$, or of $a - b\sqrt{-1}$.

98. If $M(p)$ and $M(q)$ represent respectively the moduli of p and q , then

$$M(p) \cdot M(q) = M(pq).$$

Suppose $p = a + b\sqrt{-1}$,

and $q = c + d\sqrt{-1}$.

Then $M(p) = \sqrt{a^2 + b^2}$, and $M(q) = \sqrt{c^2 + d^2}$.

Therefore $M(p) \cdot M(q) = \sqrt{(a^2 + b^2)(c^2 + d^2)} \dots \dots (1).$

But $pq = (a + b\sqrt{-1})(c + d\sqrt{-1})$

$$= (ac - bd) + (ad + bc)\sqrt{-1}.$$

$$\therefore M(pq) = \sqrt{(ac - bd)^2 + (ad + bc)^2}$$

$$= \sqrt{(a^2 + b^2)(c^2 + d^2)} \dots \dots \dots (2).$$

Hence from (1),

$$M(p) \cdot M(q) = M(pq).$$

99. *The addition, subtraction, multiplication, or division of quantities of the form $A + B\sqrt{-1}$ gives a result of the same form.*

Let $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$ be the quantities.

(i.) *Addition and subtraction.*

$$(a + b\sqrt{-1}) \pm (c + d\sqrt{-1}) = (a \pm c) + (b \pm d)\sqrt{-1}.$$

(ii.) *Multiplication.*

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}.$$

(iii.) *Division.*

$$\begin{aligned} \frac{a + b\sqrt{-1}}{c + d\sqrt{-1}} &= \frac{(a + b\sqrt{-1})(c - d\sqrt{-1})}{(c + d\sqrt{-1})(c - d\sqrt{-1})} \\ &= \frac{(ac + bd) + (bc - ad)\sqrt{-1}}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\sqrt{-1}. \end{aligned}$$

100. *To find the powers of $\sqrt{-1}$.*

Every number must be of the form $4m$, $4m+1$, $4m+2$, or $4m+3$.

$$\begin{aligned} \text{Now } (\sqrt{-1})^{4m} &= \{(\sqrt{-1})^2\}^{2m} = (-1)^{2m} = 1. \\ (\sqrt{-1})^{4m+1} &= (\sqrt{-1})^{4m} \cdot (\sqrt{-1}) = 1 \cdot \sqrt{-1} = \sqrt{-1}. \\ (\sqrt{-1})^{4m+2} &= (\sqrt{-1})^{4m} \cdot (\sqrt{-1})^2 = 1(-1) = -1. \\ (\sqrt{-1})^{4m+3} &= (\sqrt{-1})^{4m} \cdot (\sqrt{-1})^3 (\sqrt{-1}) \\ &= 1(-1)\sqrt{-1} = -\sqrt{-1}. \end{aligned}$$

101. *To find the cube roots of unity.*

These will evidently be the three values of x which satisfy the equation $x^3 = 1$, or $x^3 - 1 = 0$.

We have

$$(x - 1)(x^2 + x + 1) = 0.$$

Therefore $x - 1 = 0$, and $x^2 + x + 1 = 0$, will both furnish solutions.

From the former, $x = 1$;

And from the latter, $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.

Hence the cube roots of unity are $1, \frac{1}{2}(-1 \pm \sqrt{-3})$.

We may verify this result by *cubing* these quantities.

In each case the cube is *unity*.

Cos. Similarly, the cube roots of -1 are

$$-1, \frac{1}{2}(1 \pm \sqrt{-3}).$$

And so, the fourth roots of 1 are $\pm 1, \pm \sqrt{-1}$;

and the fourth roots of -1 are $\pm \frac{1 \pm \sqrt{-1}}{\sqrt{2}}$.

Ex. XXVI.

Express with fractional indices—

1. $\sqrt{x^3}, \sqrt[3]{a^5b^3}, \sqrt[4]{x^3y^3}, \sqrt[5]{a^2b^4}.$

2. $\frac{xy}{\sqrt{ab}}, \frac{\sqrt{c^2d}}{\sqrt{xy}}, \frac{\sqrt[3]{x^m}}{\sqrt[4]{a^n}}.$

Reduce to entire surds—

3. $3\sqrt{2}, 4\sqrt{\frac{1}{3}}, \frac{2}{3}\sqrt{15}, 3\sqrt[3]{\frac{1}{4}}.$

4. $4 \cdot 2^{\frac{1}{2}}, 9 \cdot 3^{-\frac{1}{3}}, 4 \cdot 2^{-\frac{1}{4}}, \frac{1}{2}(\frac{1}{3})^{-\frac{1}{5}}.$

5. $3\sqrt{ab}, a\sqrt{\frac{bc}{a}}, (a+x)\sqrt[3]{\frac{a-x}{(a+x)^2}}$

Reduce to a common index—

6. $\sqrt[3]{2}, \sqrt[4]{3}.$

7. $\sqrt[4]{2}, \sqrt[3]{3}.$

8. $2\sqrt{2}, 3\sqrt[3]{5}.$

9. $\sqrt[3]{a}, \sqrt[4]{b}.$

10. $(a+x)^{\frac{1}{2}}, \sqrt[3]{a-x}.$

11. $a^{\frac{1}{2}+\frac{1}{3}}, b^{\frac{1}{2}-\frac{1}{3}}.$

Simplify—

12. $\sqrt{12}, \sqrt[3]{48}, 3\sqrt{28}, \frac{1}{2}\sqrt[3]{648}.$

13. $\sqrt{4a^3+4a^2b}, \sqrt[3]{a^3b^3+b^6}, \sqrt[5]{\frac{a+b}{64a}}.$

14. $\sqrt{\frac{1}{x^2}-\frac{2a}{x}+a^2}, \sqrt[3]{\frac{a^3}{x^3}-\frac{3a^2}{x}+3a^2-a^3x}.$

15. $\sqrt[3]{\frac{x^3+1}{9x^3}+\frac{x+1}{3x}}, \sqrt[3]{\frac{(x+a)^3(x^3-a^3)}{(x-a)^3(x+a)}}.$

Find the value of—

$$16. \sqrt{12} + \sqrt{48} - 2\sqrt{3}, \sqrt{56} + \sqrt[3]{189}.$$

$$17. \sqrt{\frac{a^4x^2y^2}{b}} - \sqrt[3]{a^4b^2x^3} + \sqrt{\frac{ax^2c^2}{b}}.$$

$$18. \sqrt[3]{27a^{m+6}b^3} - \sqrt[3]{8a^{4m+9}} + 3\sqrt[3]{64a^m}.$$

Multiply—

$$19. a + \sqrt{ab} + b \text{ by } \sqrt{a} - \sqrt{b}, a^{\frac{1}{2}} + b^{\frac{1}{2}} \text{ by } \sqrt{a} - \sqrt{b}.$$

$$20. (x+y)^{\frac{1}{2}} \text{ by } (x+y)^{\frac{1}{2}}, a + b\sqrt{d} \text{ by } a^2 - ab\sqrt{d} + b^2d.$$

$$21. c\sqrt{\frac{a}{a+b}} + \sqrt{\frac{1}{b}} \text{ by } \frac{c^2}{d}\sqrt{a(a+b)} - \sqrt{b^3}.$$

$$22. a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}} + d^{\frac{1}{2}} \text{ by } a^{\frac{1}{2}} - b^{\frac{1}{2}} + c^{\frac{1}{2}} - d^{\frac{1}{2}}.$$

Divide—

$$23. x^2 + xy + y^2 \text{ by } x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y.$$

$$24. x^3 - y^3 \text{ by } x^{\frac{1}{2}} + y^{\frac{1}{2}}.$$

Rationalize the denominators of—

$$25. \sqrt[5]{3}, \frac{4}{\sqrt{7}}, \sqrt[3]{\frac{7}{5}}.$$

$$26. \frac{3}{2 + \sqrt{3}}, \frac{4}{3\sqrt{2} - 2\sqrt{3}}, \frac{1}{\sqrt{5} - \sqrt{3}}.$$

$$27. \frac{3}{1 + \sqrt{2} + \sqrt{3}}, \frac{2}{\sqrt{2} + \sqrt{3} + \sqrt{5}}, \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}.$$

$$28. \frac{a}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}, \frac{1}{\sqrt{2} + \sqrt{3}}, \frac{b}{x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y}.$$

Find the square roots of—

$$29. 11 + 4\sqrt{7}, 8 + 2\sqrt{15}, 30 - 10\sqrt{5}.$$

$$30. 8 + 2\sqrt{12}, 9 - 6\sqrt{2}, 20 - 10\sqrt{3}.$$

CHAPTER XII.

RATIO AND PROPORTION.

Ratio.

102. The ratio between two quantities is that multiple, part, or parts which the former is of the latter.

Thus, the ratio of $a : b = \frac{a}{b}$.

103. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by increasing the terms of the ratio by the same quantity.*

Let $a : b$ or $\frac{a}{b}$ be the ratio, and let each of its terms be increased by m . It will then become $\frac{a+m}{b+m}$.

Now, $\frac{a+m}{b+m} \geq \frac{a}{b}$, as $(a+m)b \geq (b+m)a$, or, as $ab + bm \geq ab + am$; or, as $bm \geq am$, or as $b \geq a$.

Hence the ratio $\frac{a}{b}$ is increased when $b > a$, that is, when it is a ratio of less inequality; and is diminished when $b < a$, that is, when it is a ratio of greater inequality.

COR. It may be shown in the same way that—

A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by diminishing the terms of the ratio by the same quantity.

104. When the difference between the antecedent and consequent is small compared with either, the ratio of the higher powers of the terms is found by doubling, trebling, &c., their difference.

Let $a + x : a$ or $\frac{a+x}{a}$ be the ratio, where x is small compared with a .

Then $\frac{(a+x)^2}{a^2} = \frac{a^2 + 2ax + x^2}{a^2} = 1 + \frac{2x}{a}$ nearly = $\frac{a+2x}{a}$ nearly.

$$\frac{(a+x)^3}{a^3} = \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a^3} = 1 + \frac{3x}{a} \text{ nearly} =$$

$$\frac{a+3x}{a} \text{ nearly; and so on.}$$

$$\text{Ex. } (1002)^3 : (1000)^3 = 1004 : 1000 \text{ nearly.}$$

$$(1002)^3 : (1000)^3 = 1006 : 1000 \text{ nearly.}$$

Proportion.

105. Proportion is the relation of equality expressed between ratios.

Thus, the expression $a : b = c : d$,

or $a : b :: c : d$,

$$\text{or } \frac{a}{b} = \frac{c}{d},$$

is called a proportion.

106. The following results are easily obtained:—

$$(1.) \text{ Since } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c} \text{ or } \frac{a}{c} = \frac{b}{d},$$

$$\therefore a : c :: b : d \text{ (alternando).}$$

$$(2.) 1 \div \frac{a}{b} = 1 \div \frac{c}{d} \text{ or } \frac{b}{a} = \frac{d}{c},$$

$$\therefore b : a :: d : c \text{ (invertendo).}$$

Also, by Art. 64, we have—

$$(3.) a + b : b :: c + d : d \text{ (componendo).}$$

$$(4.) a - b : b :: c - d : d \text{ (dividendo).}$$

$$(5.) a - b : a :: c - d : c \text{ (convertendo).}$$

$$(6.) a + b : a - b :: c + d : c - d \text{ (componendo and dividendo).}$$

107. If $a : b :: c : d$ and $e : f :: g : h$, we may compound the proportions.

$$\text{Thus we have } \frac{a}{b} = \frac{c}{d} \dots\dots (1), \text{ and } \frac{e}{f} = \frac{g}{h} \dots\dots (2).$$

$$(1) \times (2), \text{ then, } \frac{ae}{bf} = \frac{cg}{dh},$$

$$\text{or } ae : bf :: cg : dh.$$

And in the same way we may show that, if the corresponding terms of any number of proportions be multiplied together, the products will be proportional.

108. *If three quantities are in continued proportion, the first has to the third the duplicate ratio of what it has to the second.*

Let a, b, c be the given quantities in continued proportion; then—

$$\frac{a}{b} = \frac{b}{c}.$$

$$\text{Hence, } \frac{a}{c} \text{ or } \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2};$$

$$\therefore a : c :: a^2 : b^2.$$

And, similarly, if a, b, c, d are four quantities in continued proportion, $a : d :: a^3 : b^3$, that is—

The first has to the fourth the triplicate ratio of what it has to the second; and so on, for any number of quantities.

109. We shall now give one or two examples of problems in Proportion.

Ex. 1. If $a : b :: c : d$, prove that $\frac{a^3 + c^3}{b^3 + d^3} = \left(\frac{a + c}{b + d}\right)^3$.

Let $\frac{a}{b} = \frac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

Hence, $\frac{a^3 + c^3}{(a + c)^3} = \frac{(bx)^3 + (dx)^3}{(bx + dx)^3} = \frac{b^3 + d^3}{(b + d)^3}$, then, alter-

nando, $\frac{a^3 + c^3}{b^3 + d^3} = \left(\frac{a + c}{b + d}\right)^3$.

Ex. 2. If $a : b :: c : d$, prove that $\frac{a + b}{a - b} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}$.

Let $\frac{a}{b} = \frac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

$$\begin{aligned} \text{Hence, } \frac{a + b}{a - b} &= \frac{bx + b}{bx - b} = \frac{x + 1}{x - 1} = \frac{\sqrt{bd} \cdot x + \sqrt{bd}}{\sqrt{bd} \cdot x - \sqrt{bd}} \\ &= \frac{\sqrt{bx} \cdot dx + \sqrt{bd}}{\sqrt{bx} \cdot dx - \sqrt{bd}} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}. \end{aligned}$$

Or, it may be worked thus—

Since $\frac{a}{b} = \frac{c}{d}$, we have $\sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}}$

Or, $\sqrt{\frac{a}{b}} \times \sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}} \times \sqrt{\frac{c}{d}}$, or $\frac{a}{b} = \frac{\sqrt{ac}}{\sqrt{bd}}$

Hence, by Art. 106—

$$\frac{a + b}{a - b} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}$$

Ex. 3. If $a : b :: c : d :: e : f$, show that

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} \right)^{\frac{1}{r}}$$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = x \dots \dots \dots (1).$

$$\therefore \frac{a^r}{b^r} = \frac{c^r}{d^r} = \frac{e^r}{f^r} = x^r.$$

Hence, $a^r = b^r x^r, \therefore ma^r = mb^r x^r$
 $c^r = d^r x^r, \therefore nc^r = nd^r x^r$
 $e^r = f^r x^r, \therefore pe^r = pf^r x^r$ } and $\therefore$ by addition,
 $ma^r + nc^r + pe^r = (mb^r + nd^r + pf^r)x^r.$

$\therefore \frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} = x^r$, or $\left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} \right)^{\frac{1}{r}} = x \dots (2).$

$\therefore$ Equating (1) and (2), we have—

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} \right)^{\frac{1}{r}} \quad Q.E.D.$$

Ex. XXVII.

1. Compare the ratios $a + b : a - b$, and $a^2 + b^2 : a^2 - b^2$.
2. Which is the greater of the ratios $a + b : 2a$, and $2b : a + b$?
3. What quantity must be subtracted from the consequent of the ratio $a : b$ in order to make it equal to the ratio $c : d$?

4. Compound the ratios $1 - x^2 : 1 + y$, $x - xy^2 : 1 + x^2$, and $1 : x - x^2$.

5. There are two numbers in the ratio of 6 : 7, but if 10 be added to each they are in the ratio of 8 : 9. Find the numbers.

6. In what cases is $x + \frac{6}{x} >$ or < 5 ?

7. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{z-x}$, show that $a + b + c = 0$.

8. Find the value of x when the ratio $x + 2a : x + 2b$ is the duplicate ratio of $2x + a + c : 2x + b + c$.

9. Find x when the ratio $x - b : x + 2a - b$ is the triplicate ratio of $x - a : x + a - b$.

10. If $\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{x+z}{c+a}$, show that each of the fractions is equal to $\frac{x+y+z}{a+b+c}$, and that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

11. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each is equivalent to $\frac{la+mc+ne}{lb+md+nf}$.

hence, show that—

$$\frac{a}{2z + 2x - y} = \frac{b}{2x + 2y - z} = \frac{c}{2y + 2z - x}$$

$$\text{when } \frac{x}{2a + 2b - c} = \frac{y}{2b + 2c - a} = \frac{z}{2c + 2a - b}$$

12. If $a : b :: c : d$, then

$$a + b : c + d :: \sqrt{a^2 + ab + b^2} : \sqrt{c^2 + cd + d^2}.$$

13. Find a fourth proportional to the quantities—

$$\frac{x+1}{x-1}, \frac{x^2+x+1}{x^2-x+1}, \frac{x^3+1}{x^3-1}$$

14. Find c in terms of a and b when—

$$(1.) a : a :: a - b : b - c$$

$$(2.) a : b :: a - b : b - c$$

$$(3.) a : c :: a - b : b - c$$

15. If a, b, c are in continued proportion, show that $\frac{a+b}{b+c}, b, bc$ are also in continued proportion.

16. If $a : b :: c : d$, then—

$$\sqrt{a^{2n} + b^{2n}} : \sqrt{c^{2n} + d^{2n}} :: (a - b)^n : (c - d)^n.$$

17. From a vessel containing a cubic inches of hydrogen gas, b cubic inches are withdrawn, the vessel being filled up with oxygen at the same pressure. Show that if this operation be repeated n times successively, the quantity of hydrogen remaining in the vessel is $\frac{(a-b)^n}{a^{n-1}}$ cubic inches, when reduced to the original pressure.

CHAPTER XIII.

VARIATION.

110. If two quantities are connected together by some constant multiplier, either integral or fractional, one is said to *vary* as the other.

The symbol $\propto$ placed between the quantities is used to express this relation; and if we suppose m, n to be constant quantities, we may distinguish the different kinds of *variation* as follows:

- (1.) If $A = mB$, then A varies *directly* as B , or $A \propto B$.
- (2.) If $A = \frac{m}{B}$, then A varies *inversely* as B , or $A \propto \frac{1}{B}$.
- (3.) If $A = mBC$, then A varies *jointly* as B and C , or $A \propto BC$.
- (4.) If $A = m \frac{B}{C}$, then A varies *directly* as B , and *inversely* as C , or $A \propto \frac{B}{C}$.
- (5.) If $A = mB^n$, then A varies *directly* as the n th power of B , or $A \propto B^n$.

(6.) If $A = \frac{m}{B^n}$, then A varies *inversely* as the n th power of B , or $A \propto \frac{1}{B^n}$.

(7.) If $A = mB + \frac{n}{C}$, then A varies as the sum of two quantities, one of which varies *directly* as B , and the other *inversely* as C .

111. If $A \propto B$, and $B \propto C$, then $A \propto C$.

For, let $A = mB$ (1),
and $B = nC$;

then, substituting in (1) for this value of B , we have

$$A = m \cdot nC = mnC; \text{ or, since } mn \text{ is constant,} \\ A \propto C. \quad Q.E.D.$$

112. If $A \propto B$, then will $A^2B + AB^2 \propto (A^2 + B^2)(A - B)$.

For, let $A = mB$, then

$$A^2B + AB^2 = (mB)^2 \cdot B + mB \cdot B^2 = (m^2 + m) B^3 \dots (1).$$

$$\text{And so, } (A^2 + B^2)(A - B) = (m^2 + 1)(m - 1) \cdot B^3 \dots (2).$$

$$(1) \div (2), \text{ then } \frac{A^2B + AB^2}{(A^2 + B^2)(A - B)} = \frac{m^2 + m}{(m^2 + 1)(m - 1)}, \text{ a} \\ \text{constant quantity.}$$

$$\therefore A^2B + AB^2 \propto (A^2 + B^2)(A - B). \quad Q.E.D.$$

113. If $A \propto B$, when C is constant,
and $A \propto C$, when B is constant;
then $A \propto BC$, when BOTH are VARIABLE.

Let a, b be corresponding values of A and B , when C is constant,

$$\text{Then} \quad A : a, :: B : b, \text{ or } a, = \frac{Ab}{B} \dots \dots \dots (1).$$

Now this value a , of A will, like A itself, vary as C .

Let then a, c be corresponding values of a, C , when B is constant.

$$\text{Then} \quad a, : a :: C : c, \text{ or } a, = \frac{Ca}{c} \dots \dots \dots (2).$$

(1) = (2), then

$$\frac{Ab}{B} = \frac{Ca}{c}, \text{ or } A = \frac{a}{bc} BC.$$

Hence A bears a constant ratio $\frac{a}{bc}$ to the product of the variable quantities B and C.

$\therefore A \propto BC$, when both B and C are variable. *Q.E.D.*

This proposition is the foundation of the so-called Double Rule of Three in Arithmetic.

We will illustrate this by an example,

Ex. If 15 men earn £180 in 6 weeks, how much will 22 men earn in 13 weeks?

We have ·

Number of pounds $\propto$ number of men, when time is not considered; that is, when number of weeks is constant.

Hence,

15 : 22 :: 180 : number of pounds required, when *time* is not considered.

Again,

Number of pounds $\propto$ number of weeks, when *men* are not considered; that is, when number of men is constant.

Hence,

6 : 13 :: 180 : number of pounds required, when *men* are not considered.

These two statements correspond to the ordinary statement of a Double Rule of Three sum.

But, by the above proposition :

Number of pounds $\propto$ number of men $\times$ number of weeks, when *both* are considered.

Hence,

15 $\times$ 6 : 22 $\times$ 13 :: 180 : number of pounds required.

This is the ordinary complete statement.

COR. If A varies as each of the quantities B, C, D, &c.,

when all the rest are constant, then when all are variable, A varies as their product.

Ex. XXVIII.

1. If $x \propto y$, and when $x = 3$, $y = 7$; find the value of x when $y = 63$.

2. If $A \propto BC$, and that $C = 3$, when $A = 60$, and $B = 1$; find the value of B , when $A = 80$, and $C = 4$.

3. If $x = my^2 + nx^2$, find m , n , when 17, 2, 1, and 23, 1, 2, are corresponding sets of values of x , y , z .

4. If a cylinder whose length is l , and diameter d , has a weight w , what will be the weight of a cylinder of the same material whose length is l , and diameter d ?

5. If t be the time of a complete vibration of a pendulum of length l , then $t \propto \sqrt{l}$. Hence, find the length of a two-second pendulum, when the length of the second pendulum is 39.4 inches.

6. The time taken by a falling body varies as the square root of the space. Now, a body falls through 64 feet in 2 seconds. Hence, find the relation between time and space.

7. The strength of a beam supported at its ends varies inversely as its length, directly as its breadth, and directly as the square of its depth. Now, a beam whose length, breadth, and thickness are respectively 12 feet, 6 inches, 4 inches, will support 3 tons. Find the breaking-weight of a beam of the same material whose corresponding dimensions are 16 feet, 8 inches, 6 inches.

8. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver.

9. The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ the cube of their weight. A diamond of a carats is worth m times a ruby of b carats, and both together are worth c £. Required the value of a diamond and a ruby, each weighing n carats.

10. The heat generated by the discharge of a Leyden jar varies jointly as the *quantity* and the *tension* of the electricity; and the tension varies *directly* as the *quantity* of electricity, and *inversely* as the *surface* of the jar. Show that when the surface is constant, the quantity of electricity varies directly as the square root of the heat.

11. A rod of iron weighs 10 lbs. per foot run, what is the length in metres per kilogram?

12. British units of length (L), mass (M), and time (T) are respectively a foot, a pound, and a second; and the corresponding absolute units are a metre, a kilogram, and a second. Now, the dimensions of the units of force and work are respectively $\frac{LM}{T^2}$ and $\frac{L^2M}{T^2}$. Hence express the absolute units of force and work as British units.

CHAPTER XIV.

PERMUTATIONS AND COMBINATIONS.

Permutations.

114. The different arrangements that can be made of a number of things are called their **Permutations**.

Thus, *ab, ac, bc, ba, ca, cb*, are called the permutations of *a, b, c*, taken *two together*; and *abc, acb, bca, bac, cab, cba*, are the permutations of the same things taken *three together*.

In some works, the term *permutations* is limited to the arrangements of the things when taken *all together*. When they are *not taken all together*, they are called **Variations**. We shall include under the term "permutations" all such arrangements.

115. The *different groups* that can be made of a number of things *without* regard to the order of the things are called **Combinations**.

Thus ab, ba form the same combination; so that the combinations of a, b, c taken two together are ab, ac, bc .

116. *The number of permutations of n things taken r together is*

$$n(n-1)(n-2)\dots(n-r+1).$$

Let $a, b, c, \dots, k$ be the n things. We will first take them two together. Now, there must be $n-1$ things before which a stands first, and the same number before which b stands first, and so on for all the letters. Hence the total number of arrangements must be n times $(n-1)$, or $n(n-1)$. And hence the number of permutations of n things taken two together $= n(n-1)$.

Again, if we first consider the $(n-1)$ things $b, c, \dots, k$, we have just shown—as may be seen by putting $(n-1)$ for n in the above result—that

Number of permutations of $(n-1)$ things taken two together $= (n-1)(n-1-1) = (n-1)(n-2)$.

Now, a may stand before each of these permutations of $b, c, \dots, k$.

Hence there are $(n-1)(n-2)$ permutations in which a stands first, when the n letters $a, b, c, \dots, k$ are permuted three together. And, of course, there are just as many in which $b, c, \dots, k$ each stand first.

Hence, the whole number of permutations of the n things taken three together $= n$ times $(n-1)(n-2) =$
 $n(n-1)(n-2)$.

And by similar reasoning we may show that the number of permutations of n things taken four together $=$
 $n(n-1)(n-2)(n-3)$.

If we look at the last term of the last factor of each of these results we see that its value (neglecting the sign) is one less than the number of things taken together.

It is easy then to perceive an apparent general law, viz.:
 Number of permutations of n things taken r together

$$= n(n-1)(n-2)\dots(n-r+1).$$

By what is called *Mathematical Induction* we may assure ourselves of the truth of this law.

We will *assume* the law to be true when p things are taken together, and will then show that it must be true when $(p + 1)$ things are taken together.

Fixing our mind upon $(n - 1)$ things, viz., $b, c, \dots k$, we have by our assumption—putting therein $(n - 1)$ for n ,

Number of permutations of $b, c, \dots k$, taken p together

$$= (n - 1)(n - 2) \dots (n - 1 - \overline{p - 1}) \\ = (n - 1)(n - 2) \dots (n - p).$$

Now, a may stand before each of these, and hence the number of permutations of n things taken $(p + 1)$ together, where a stands first, is $(n - 1)(n - 2) \dots (n - p)$.

And the same may be said for each of the letters $b, c, \dots k$.

Hence the whole number of permutations of n things taken $(p + 1)$ together is

$$n \text{ times } (n - 1)(n - 2) \dots (n - p),$$

$$\text{or } n(n - 1)(n - 2) \dots \{n - (\overline{p + 1} - 1)\}.$$

Hence, *on the assumption that the law is true for any particular case*, we have shown it to be also true when the things are taken *one more together*.

Now, we know it to be true when the things are taken 2, 3, 4 together. It is therefore true when taken 5 together, and therefore also when taken 6, 7, &c., together. Hence, it is generally true.

We have therefore generally

Number of permutations of n things taken r together

$$= n(n - 1) \dots (n - \overline{r - 1}), \text{ or} \\ = n(n - 1) \dots (n - r + 1).$$

COR. 1. Let $r = n$, then the whole number of permutations of n things taken *all* together $= n(n - 1) \dots (n - n + 1)$

$$= n(n - 1) \dots 1, \text{ or, as often written,} \\ = \lfloor n, \text{ read factorial } n.$$

COR. 2. Every *combination* of r things will produce $\lfloor r$ permutations, for the number is that of r things taken all together.

117. *To find the number of permutations of n things taken all together when they are not all different things.*

Let there be n letters, in which the letter a occurs p times, b occurs q times, c occurs r times, &c. Then the number of permutations of the n things when taken altogether is

$$\frac{n}{p \ q \ r \ \&c.}$$

For (Art. 116, Cor. 1) the number of permutations of n different things taken all together = n .

Let P be the required number of permutations. Then, since every permutation contains a repeated p times, it is capable of forming p permutations on the supposition that the a 's are all different letters. The whole number of permutations would then be P times p , or $P \cdot p$. And similarly, on the supposition that all the b 's are different, every permutation would form q permutations, and so on.

Hence, on the supposition that all the letters are different, the total number of permutations = $P \cdot p \cdot q \cdot r \ \&c.$

Hence we have $P \cdot p \cdot q \cdot r \ \&c. = n$,

And $\therefore P = \frac{n}{p \cdot q \cdot r \ \&c.}$

118. *To find the number of permutations of n things when each may be repeated once, twice, &c., times.*

Let $a, b, c, \dots k$ be the n quantities.

When taken one at a time, their number is n .

If we place a before each of these letters, we get n permutations $aa, ab, ac, \dots ak$, in which a stands first; and, we must also have as many where b stands first, where c stands first, &c.

Hence, on the whole, when the letters are taken two together, the number of permutations = n times $n = n^2$.

And so we may show that, when taken three together, the number of permutations = n times $n^2 = n^3$.

Hence it is easy to show by induction that the number

of permutations of n things when each may be repeated r time = n^r .

Ex. 1. Find the number of permutations of 8 things taken 4 together.

Generally (Art. 13) $P = n(n-1) \dots (n-r+1)$.
 Here $n = 8$ } $\therefore P = 8(8-1) \dots (8-4+1) = 8.7.6.5$.
 $r = 4$

Ex. 2. Find the number of permutations which can be made out of the letters of the word *Wagga-Wagga* when taken all together.

$$\text{Generally (Art. 117) } P = \frac{|n|}{|p| |q| |r|}$$

Here $n = 10$, and, if p, q, r represent respectively the number of times w, a, g appear,

$$p = 2, q = 4, r = 4.$$

$$\text{Hence } P = \frac{|10|}{|2| |4| |4|} = 5.7.9.10.$$

Combinations.

119. *To find the number of combinations of n things taken r together.*

Let C_r be the required number of combinations.

Now (Art. 13, Cor. 2) every combination of r things is capable of forming $|r|$ permutations.

Hence the whole number of permutations capable of being formed by C_r combinations = $|r|$ times $C_r = |r| C_r$.

Now this is the number of permutations of n things taken r together.

Hence, by Art. 116

$$|r| C_r = n(n-1) \dots (n-r+1).$$

$$\therefore C_r = \frac{n(n-1) \dots (n-r+1)}{|r|}.$$

COR. Since

$$\begin{aligned}
 & \frac{n(n-1) \dots (n-r+1)}{[r]} \\
 &= \frac{n(n-1) \dots (n-r+1) \cdot (n-r) \dots 2 \cdot 1}{[r] \cdot [n-r]} \\
 &= \frac{[n]}{[r] [n-r]}, \text{ we have} \\
 C_r &= \frac{[n]}{[r] [n-r]}.
 \end{aligned}$$

120. To show that $C_r = C_{n-r}$ or that the number of combinations of n things taken r together is equal to the number taken $(n-r)$ together.

We have

$$C_r = \frac{n(n-1) \dots (n-r+1)}{[r]},$$

and therefore putting $n-r$ for r ,

$$\begin{aligned}
 C_{n-r} &= \frac{n(n-1) \dots (n-n-r+1)}{[n-r]} \\
 &= \frac{n(n-1) \dots (r+1)}{[n-r]} \\
 &= \frac{n(n-1) \dots (r+1) \cdot r(r-1) \dots 2 \cdot 1}{[n-r] \cdot [r]} \\
 &= \frac{[n]}{[n-r] \cdot [r]}.
 \end{aligned}$$

Hence (Art. 119, Cor.) $C_r = C_{n-r}$ Q.E.D.

121. To find the number of combinations that can be formed of n sets of things, in which there are p of one sort, q of another, r of another, &c., by taking one from each set.

First, if there be two sets, p of one sort and q of another, we may place each of the former sort before each of the latter, and thus obtain p times q of combinations.

The number of combinations is therefore pq .

Again, if there be a third set containing r things, we may

place each one of this set before all the above combinations in turn.

Hence the total number of combinations thus formed is r times pq , or pqr .

And so on for any number of sets.

Hence, number of combinations required = $pqr \dots$

122. *To find the number of combinations of m things and n things when p of the former are combined with q of the latter.*

The number of combinations of m things taken p together

$$= \frac{m(m-1) \dots (m-p+1)}{|p|}.$$

And the number of combinations of n things taken q together

$$= \frac{n(n-1) \dots (n-q+1)}{|q|}.$$

Now each of the former combinations may be combined in turn with all of the latter. Hence the total number of combinations

$$= \frac{m(m-1) \dots (m-p+1)}{|p|} \cdot \frac{n(n-1) \dots (n-q+1)}{|q|}.$$

123. *To find when the number of combinations is greatest.*

$$\text{We have } C_r = \frac{n(n-1) \dots (n-r+1)}{|r|},$$

$$\begin{aligned} \text{and } C_{r-1} &= \frac{n(n-1) \dots (n-r+1+1)}{|r-1|} \\ &= \frac{n(n-1) \dots (n-r+2)}{|r-1|}. \end{aligned}$$

$$\text{Hence, by division, } \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = \frac{n+1}{r} - 1.$$

Now as r continually increases, $\left(\frac{n+1}{r} - 1\right)$ continually diminishes, and therefore must eventually become *less than unity*.

But as long as its value is > 1 , the number of combinations found must be greater than the number previously found. Hence the greatest number of combinations will be for that value of r next before the one which makes $\left(\frac{n+1}{r} - 1\right)$ less than unity.

When $\frac{n+1}{r} - 1 = 1$, of course $C_r = C_{r-1}$; and since the next higher value of r will make $\left(\frac{n+1}{r} - 1\right) < 1$, the value of r obtained from this equation will give the greatest number of combinations.

Solving the equation, we have $r = \left(\frac{n+1}{2}\right)$; and, since r is integral, n must be odd.

Hence, when n is odd,

The greatest number of combinations is when taken $\frac{n+1}{2}$ together, and also when taken $\frac{n+1}{2} - 1$, or $\frac{n-1}{2}$ together.

When n is even, we require the value of r next before the one which makes $\frac{n+1}{r} - 1 < 1$, or $\frac{n+1}{r} < 2$, or $n+1 < 2r$, or $r > \frac{n+1}{2}$.

The value of r required is $\therefore r = \frac{n}{2}$.

Hence, when n is even,

The greatest number of combinations is when they are taken $\frac{n}{2}$ together.

Ex. 1. Find the number of combinations of 7 things taken 3 together.

$$\text{Number required} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

Ex. 2. How many words can be formed out of 19 consonants and 7 vowels by taking 4 consonants and 2 vowels for each word?

Number of combinations formed by 19 consonants taken 4 together

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4}.$$

And number formed by 7 vowels taken 2 together = $\frac{7 \cdot 6}{1 \cdot 2}$.

Hence (Art. 122),

Number of combinations by taking 4 consonants and 2 vowels.

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{7 \cdot 6}{1 \cdot 2} = 81396.$$

Now each combination contains 6 letters, and these may (Art. 116) be permuted in 6 ways.

Hence total number of words required = $81396 \times \underline{6}$.
= 58605120.

Ex. XXIX.

1. The number of permutations of some things taken 5 together : the number taken 3 together :: 42 : 1. Find the number of things.

2. How many different permutations can be made of the letters of the word *Proportion* taken all together?

3. How many can be made of each of the words *Kilimaud-jaro*, *Hammada*?

4. Show that the number of triangles which may be formed by joining the angular points of a polygon is $\frac{1}{2} n(n-1)(n-2)$, n being the number of sides.

5. How many different numbers can be made by using all the figures of the number 302342044?

6. In how many ways can 10 persons be arranged about a round table?

7. The number of combinations of a number of things taken 5 together is the same as the number taken 7 together. Find the number of things.

8. The number of combinations of n things taken 2 together exceeds by 6 the number of combinations of $(n - 1)$ things taken two together. Find n .

9. If P_r represent the number of permutations of n things taken r together, show that, when $m > 2$,

$$(P_1 - 1)(P_2 - P_1)(P_3 - P_2)(P_4 - P_3) \dots (P_m - P_{m-1}) \\ = P_2 P_3 P_4 \dots P_{m-1} \cdot P_{m+1}.$$

10. The number of combinations of $2n$ things taken n together

$$= 2^n \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n - 1)}{1 \cdot 2 \cdot 3 \dots n}$$

11. The ratio of the combinations of $4n$ things taken $2n$ together to the combinations of $2n$ things taken n together

$$= \frac{1 \cdot 3 \cdot 5 \dots (4n - 1)}{(1 \cdot 3 \cdot 5 \dots 2n - 1)^2}.$$

12. Find the *total number* of signals which can be made by m needles, each of which can assume n distinct positions.

13. Show that $C_r^2 = \left(1 + \frac{1}{r}\right) \left(1 + \frac{1}{n-r}\right) C_{r-1} \cdot C_{r+1}$.

14. If ${}^nP_r$ represent the number of permutations of n things taken r together, show that ${}^{n+1}P_r - {}^nP_r = r {}^nP_{r-1}$.

15. Show that $2P_2 + 3P_3 + \&c. + rP_r \\ = (n - 1)(P_1 + P_2 + \&c. + P_r) - P_{r+1}.$

16. At a Parliamentary election for two members there were four candidates. In how many ways can votes be recorded?

17. A firing party, consisting of one sergeant and 12 men, is to be chosen from a company of 100 men and 3 sergeants. In how many ways can this be done?

18. In taking a handful of shot from a bag, show that the chance of getting an even number is less than the chance of getting an odd number.

(The truth of the Binomial Theorem may be assumed.)

CHAPTER XV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

Arithmetical Progression:

124. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a common difference.

Thus, the numbers 1, 3, 5, 7, &c., are in A. P., and so are the numbers 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$, &c.

In the first series the common difference is 2, and in the second $-\frac{1}{2}$.

If a be the first term, and d the common difference, the series is evidently $a, a + d, a + 2d, a + 3d$, &c., and the n th term is $a + n - 1 \cdot d$.

When the series contains n terms, the n th term is the *last* term. Hence, representing the last term by l , we have

$$l = a + n - 1 \cdot d.$$

125. *To find the sum of n terms of an A. P.*

Let a be the first term, d the common difference, and S the required sum ;

$$\text{Then } S = a + \overline{a + d} + \overline{a + 2d} + \dots + l \dots (1).$$

And, writing the series in reverse order, and remembering that the terms will then diminish by d , we have

$$S = l + \overline{l - d} + \overline{l - 2d} + \dots + a \dots (2).$$

Then adding together (1) and (2) we have

$$2S = \overline{a + l} + \overline{a + l} + \overline{a + l} + \dots + \overline{a + l};$$

and, since there are n of the terms $(a + l)$, we have

$$2S = (a + l) n, \text{ or } S = (a + l) \frac{n}{2}.$$

Hence we have the following practical rule to find the sum :

Add together the first and last terms, and multiply the result by half the number of terms.

Cor. Since (Art. 124), $l = a + \overline{n-1} \cdot d$, we have

$$S = (a + a + \overline{n-1} \cdot d) \frac{n}{2} = (2a + \overline{n-1} \cdot d) \frac{n}{2};$$

which expresses the sum of an A. P. in terms of the first term, the common difference, and the number of terms.

126. To insert *m* arithmetic means between *a* and *b*.

As there are *m* means, there must be altogether *m* + 2 terms.

Hence (Art. 124),

$$l = a + \overline{(m+2)-1} \cdot d = a + \overline{m+1} \cdot d.$$

Here $l = b$;

$$\text{and therefore } a + \overline{m+1} \cdot d = b, \text{ from which } d = \frac{b-a}{m+1}.$$

Hence the means required are

$$a + \frac{b-a}{m+1}, a + 2 \cdot \frac{b-a}{m+1}, \&c., a + m \cdot \frac{b-a}{m+1};$$

$$\text{or } \frac{ma+b}{m+1}, \frac{m-1 \cdot a + 2b}{m+1}, \&c., \frac{a+mb}{m+1}.$$

Ex. 1. Find the sum of 16 terms of the series 2, 4, 6, &c.

$$\text{Generally, } S = (2a + \overline{n-1} \cdot d) \frac{n}{2}.$$

$$\text{Here } \left. \begin{array}{l} a = 2 \\ d = 2 \\ n = 16 \end{array} \right\}$$

$$\therefore S = \left\{ 2(2) + \overline{16-1} \cdot 2 \right\} \frac{16}{2} = (4 + 30) 8 = 272.$$

Or thus, $l = a + \overline{n-1} \cdot d = 2 + \overline{16-1} \cdot 2 = 32.$

Hence, by Rule (Art. 125),

$$S = (2 + 32) \frac{16}{2} = 272, \text{ as before.}$$

Ex. 2. Sum the series 12, $11\frac{1}{4}$, $10\frac{1}{2}$, &c., to 8 terms.

$$\text{We have } S = (2a + \overline{n-1} \cdot d) \frac{n}{2}.$$

$$\text{Here } \left. \begin{array}{l} a = 12 \\ d = -\frac{3}{4} \\ n = 8 \end{array} \right\}$$

$$\therefore S = \left\{ 2(12) + \overline{8-1} \left(-\frac{3}{4}\right) \right\} \frac{8}{2} = \left(24 - \frac{21}{4}\right) 4 = 75.$$

Ex. 3. Insert 8 arithmetical means between 3 and 21.

$$\text{We have, Art. 126, } d = \frac{b-a}{m+1}.$$

$$\text{Here } \left. \begin{array}{l} a = 3 \\ b = 21 \\ m = 8 \end{array} \right\}$$

$$\therefore d = \frac{21-3}{8+1} = \frac{18}{9} = 2, \text{ the common difference.}$$

And since the first term is 3, the 8 means are 5, 7, 9, 11, 13, 15, 17, 19.

Ex. 4. The sum of an A. P. is 140, the first term is 5, and common difference 2; find the number of terms.

$$\text{We have } S = (2a + \overline{n-1} \cdot d) \frac{n}{2}.$$

Here

$$\left. \begin{array}{l} S = 140 \\ a = 5 \\ d = 2 \end{array} \right\} \text{Hence } 140 = \left\{ 2(5) + \overline{n-1} \cdot 2 \right\} \frac{n}{2} = (n+4)n,$$

$$\begin{aligned} \text{or, transposing, } n^2 + 4n - 140 &= 0, \text{ or} \\ (n-10)(n+14) &= 0; \text{ from which} \\ n &= 10 \text{ or } -14. \end{aligned}$$

The value $n=10$ tells us that, commencing from the first term of the series and counting 10 terms, the sum of the terms is 140. The value $n=-14$ tells that, commencing from the *opposite end* of the series, and writing down 14 terms, the sum of the terms is also 140. The following would be the two series:

5, 7, 9, 11, 13, 15, 17, 19, 21, 23;
and -3, -1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23.

When the two values of n are *both* positive and integral,

the series must in both cases be commenced at the first term. It will always be found that the *extra* terms which are given by the larger value will of themselves give *zero* for a sum.

Should one of the values of n be fractional, we may still give it an interpretation. This will be best understood from the following example :

Ex. 5. The sum of an A. P. is 270, the first term 9, and the common difference 4 ; find the number of terms.

$$\text{We have} \quad S = (2a + \overline{n-1} \cdot d) \frac{n}{2}$$

Here

$$\left. \begin{array}{l} S = 270 \\ a = 9 \\ d = 4 \end{array} \right\}, \therefore 270 = \left\{ 2(9) + \overline{n-1} \cdot 4 \right\} \frac{n}{2} = 2n^2 + 7n$$

Transposing, then $2n^2 + 7n - 270 = 0$, or
 $(n - 10)(2n + 27) = 0$; from which
 $n = 10$ or $-13\frac{1}{2}$.

The first value $n = 10$ gives the series
 9, 13, 17, 21, &c., 45.

The second value, $n = -13\frac{1}{2}$, being negative, tells us to commence with the *opposite end* of the series, viz., the term 45, and write down 13 complete terms, and *another term* which is made up of half the sum of the 13th term and *three-fourths the common difference*. The 13 complete terms beginning with the term 45, and writing from right to left, are

- 3, 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45.

The *incomplete term* is $\frac{1}{2} \{ -3 + \frac{3}{4}(-4) \} = -3$.

And the sum of the numbers - 3, - 3, 1, 5 is *zero*, agreeably to the remark made in Ex. 4 above.

Ex. XXX.

1. Find the 10th term of the series 2, 6, 10, &c., and the sum to 12 terms.
2. Find the last term of the series 3, $3\frac{1}{2}$, $4\frac{1}{2}$, &c., to 20 terms.

Sum the following series :

3. 10, 8, 6, &c., to 9 terms.
4. $5\frac{1}{2}$, $6\frac{1}{2}$, $7\frac{1}{2}$, &c., to 12 terms.
5. 1, 3, 5, 7, &c., to 50 terms.
6. $a + 6d$, $a + 5d$, $a + 4d$, &c., to 13 terms.
7. $x + 9y$, $x + 7y$, $x + 5y$, &c., to 10 terms.
8. $3\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{1}{2}$, &c., to 14 terms.
9. Insert 12 arithmetic means between 2 and 41.
10. Find the sum of 12 terms of the series whose 5th term is 14, and 10th term 31.
11. The sum of 12 terms of an A. P. is 123, and the common difference is $1\frac{1}{2}$; find the first term.
12. Given m the middle term of an A. P., and n the number of terms; show that the sum of the series is mn .
13. If P, Q be the p th and q th terms respectively, find the first term and the common difference.
14. If S_r represent the sum of r terms of an A. P., show that

$$S_{2n+1} = S_{n+1} + S_{n-1} + (n^2 + n - 2)d.$$

15. If P, Q, R be respectively the p th, q th, r th terms, show that

$$P(q - r) + Q(r - p) + R(p - q) = 0.$$

16. There are m arithmetical progressions, the first term of each of which is 1, and the common differences respectively 1, 2, 3, &c., m . Prove that the sum of the n th terms

$$= \frac{n-1 \cdot m^2 + n+1 \cdot m}{2}.$$

17. If $\phi(n)$ represent the sum of n terms of an A. P., show that

$$\phi(3n) = 3 \left\{ \phi(2n) - \phi(n) \right\}.$$

18. Show that $\phi(n) + \phi(n+1) + \phi(n+2) + \dots + \phi(2n)$

$$= \frac{n(3n-1)}{1 \cdot 2} a + \frac{n(n-1)(7n-2)}{1 \cdot 2 \cdot 3} d.$$

19. Show also that

$$\frac{\phi(n)}{n} + \frac{\phi(n-1)}{n-1} + \&c., \text{ to } n \text{ terms} = na + \frac{1}{2}n(n-1)d.$$

20. Find the sum of n terms of the series whose r th term is $3r - 5$.

21. The sum of three numbers in A. P. is 15, and the sum of their squares is 83. Find the numbers.

22. The sum of four numbers in A. P. is 10, and the sum of their products taken two together is 35. Find the numbers.

23. The result in Ex. 15 may be arranged thus:

$$p(Q - R) + q(R - P) + r(P - Q) = 0.$$

What A. P. does this suggest; and what is the relation between the common difference of this series and the one in Ex. 15?

24. How many terms of the series 4, 7, 10, &c., amount to 144?

Geometrical Progression.

127. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *common ratio*.

Thus 2, 6, 18, 54, &c., is a G. P., and so is 12, 4, $1\frac{1}{3}$, &c., the common ratios being 3 and $\frac{1}{3}$ respectively.

It is evident that the common ratio may be found by dividing any term by the *preceding term*.

Let a be the first term, and r the common ratio.

The G. P. would then be

$$a, ar, ar^2, ar^3, \&c.$$

If there be n terms, the last or n th term $= ar^{n-1}$.

Hence we have in G. P., $l = ar^{n-1}$ (1).

128. *To find the sum of a Geometrical Progression.*

Let a be the first term, r the common ratio, and S the sum of n terms; then we have

$$S = a + ar + ar^2 + \&c., + ar^{n-1},$$

or multiplying each side by r ,

$$rS = ar + ar^2 + \&c., + ar^{n-1} + ar^n.$$

Hence, subtracting the upper line from the lower, we have

$$rS - S = ar^n - a, \text{ or } S(r - 1) = a(r^n - 1);$$

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} \dots\dots\dots (2).$$

$$\text{Since } a \cdot \frac{r^n - 1}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{r \cdot ar^{n-1} - a}{r - 1} = \frac{rl - a}{r - 1},$$

$$\text{We have also, } S = \frac{rl - a}{r - 1} \dots\dots\dots (3).$$

When r is less than unity, it is usual to write these results as follows, being obtained by changing the signs of numerator and denominator in each case and reversing the order ;

$$S = a \cdot \frac{1 - r^n}{1 - r} = \frac{a - rl}{1 - r} \dots\dots\dots (4).$$

129. *To find the limit of an infinite G. P. whose common ratio is less than unity.*

DEF. The Limit of a series is that quantity to which the sum of the series continually approaches as the number of terms increases.

$$\text{We have (Art. 128), } S = a \cdot \frac{1 - r^n}{1 - r}.$$

But since $r < 1$, and n indefinitely large, the value of r^n is indefinitely small, and becomes smaller and smaller the larger n is. Indeed, the value of r^n continually approaches zero, as n increases indefinitely.

Let Σ represent the limit of the sum of the series, then we have

$$\Sigma = a \cdot \frac{1 - 0}{1 - r} = \frac{a}{1 - r}.$$

130. *To insert m geometrical means between a and b .*

Since there are m means, there must be $m + 2$ terms.

$$\text{Hence (Art. 127) } l = ar^{m+2-1} = ar^{m+1}.$$

And therefore, since here $l = b$, we have

$$ar^{m+1} = b, \text{ or } r^{m+1} = \frac{b}{a}.$$

$$\therefore r = \sqrt[m+1]{\frac{b}{a}}.$$

This gives the common ratio, and the means can therefore be easily inserted.

Ex. 1. Sum the series 2, 4, 8, 16, &c., to 10 terms.

$$\text{Generally, } S = a \cdot \frac{r^n - 1}{r - 1}.$$

$$\left. \begin{array}{l} \text{Here } a = 2 \\ r = 2 \\ n = 10 \end{array} \right\} \therefore S = 2 \cdot \frac{2^{10} - 1}{2 - 1} = 2(2^{10} - 1).$$

Ex. 2. Given the first term of a G. P. to be 5, the last term 160, and the number of terms 6. Find the common ratio.

By Art. 127, $l = ar^{n-1}$.

$$\left. \begin{array}{l} \text{Here } l = 160 \\ a = 5 \\ n = 6 \end{array} \right\} \text{Hence, substituting, } 160 = 5r^{6-1} = 5r^5 \\ \therefore r^5 = 160 \div 5 = 32, \text{ or } r = \sqrt[5]{32} = 2.$$

Ex. 3. Find the value of the recurring decimal .666 &c.

The value required is the limit of the G. P. $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \dots$ to infinity.

$$\text{Now (Art. 129) } \Sigma = \frac{a}{1 - r}.$$

$$\left. \begin{array}{l} \text{Here } a = \frac{6}{10} \\ r = \frac{1}{10} \end{array} \right\} \therefore \Sigma = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{9} = \frac{2}{3}.$$

The meaning of this result is that $\frac{2}{3}$ is the quantity to which the value of .666 &c. approaches as the number of decimal figures taken is increased.

Ex. 4. Find the sum to n terms, and the limit to infinity, of the series:

$$a + (a + b)r + (a + 2b)r^2 + \dots, \text{ when } r < 1.$$

We have

$$S = a + (a+b)r + (a+2b)r^2 + \&c. + (a + \overline{n-1}.b)r^{n-1}$$

$$\text{And } \therefore Sr = ar + (a+b)r^2 + \&c. + (a + \overline{n-2}.b)r^{n-1} + (a + \overline{n-1}.b)r^n.$$

Therefore, subtracting,

$$S(1-r) = a + br + br^2 + \&c. + br^{n-1} - (a + \overline{n-1}.b)r^n.$$

But (Art. 128) the sum of the G. P.

$$br + br^2 + \&c. + br^{n-1} = br \cdot \frac{1-r^{n-1}}{1-r}.$$

$$\therefore S(1-r) = a + br \cdot \frac{1-r^{n-1}}{1-r} - (a + \overline{n-1}.b)r^n.$$

$$\therefore S = \frac{a}{1-r} + br \cdot \frac{1-r^{n-1}}{(1-r)^2} - (a + \overline{n-1}.b) \cdot \frac{r^n}{1-r} \dots (1).$$

If r be a proper fraction, the limits of r^{n-1} and r^n , as n indefinitely increases, are evidently zero.

Hence we have

$$\begin{aligned} S &= \frac{a}{1-r} + br \cdot \frac{1-0}{(1-r)^2} - (a + \overline{n-1}.b) \cdot \frac{0}{1-r} \\ &= \frac{a}{1-r} + \frac{br}{(1-r)^2}. \end{aligned}$$

Ex. XXXI.

1. Find the 6th term of the series 3, 6, 12, &c.
2. Find the 5th term of the series 35, 7, $1\frac{1}{2}$, &c.
3. Sum the series 81, - 27, 9, &c., to 6 terms.
4. Sum the series $2\frac{1}{3}$, 7, $17\frac{1}{3}$, &c., to 12 terms.
5. Sum the series 6, - 2, $\frac{2}{3}$, &c., to n terms.
6. Find the sum of 10 terms of the series whose first term is $\frac{1}{2}$, and third term $\frac{1}{8}$.

Sum to infinity the following series:

7. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$
8. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \&c.$
9. $10 - 9 + 8 - 7 + 6 - 5 + \&c.$
10. $5\frac{1}{2} + 1\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c.$
11. $\cdot 6 - \cdot 06 + \cdot 006 - \&c.$

$$12. a^2b - a + \frac{1}{b} - \&c.$$

$$13. (a^2 - b^2) + (a - b) + \frac{a - b}{a + b} + \&c.$$

$$14. (\sqrt{5} + 2) + 1 + (\sqrt{5} - 2) + (9 - 4\sqrt{5}) + \&c.$$

$$15. 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \&c.$$

$$16. \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \&c.$$

$$17. \frac{\sqrt{3} + 1}{\sqrt{3} - 1} + 1 + \frac{\sqrt{3} - 1}{\sqrt{3} + 1} + \&c.$$

$$18. \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \&c.$$

19. If the sum of 10 terms of G. P. is 513, and common ratio is -2 , find the series.

20. Find the 5th term of the G. P. whose 3rd and 6th terms are respectively a, b .

21. If P, Q, R be respectively the p th, q th, r th terms of a geometrical series, then

$$P^{r-q} \cdot Q^{r-p} \cdot R^{p-q} = 1.$$

22. Show that the sum of 2^n terms of a G. P. series may be expressed in the form

$$a(r + 1)(r^2 + 1)(r^4 + 1) \dots (r^{2^{n-1}} + 1).$$

23. If s_1 be the sum of an infinite series, and s_2 the sum of the squares of the terms of the series, find the series.

24. If s_p be the sum of p terms of a geometric progression, show that

$$s_n - s_m = r^m s_n - r^n s_m.$$

25. If s_m be the sum of the m th powers of an infinite geometric series, then

$$\frac{1}{s_1} \pm \frac{1}{s_2} \pm \frac{1}{s_3} \pm \dots \text{ad. inf.} = \frac{1}{a \pm 1} - \frac{r}{a \pm r}.$$

26. If $\frac{ma + nb}{ma - nb} = \frac{mb + nc}{mb - nc} = \frac{mc + nd}{mc - nd} = \&c.$, then $a, b, c, d, \&c.$, are in G. P.

27. If s_p be the sum of p terms of a G. P., show that the sum of the products of every two terms $= \frac{r}{r+1} s_p \cdot s_p - 1$.

28. Show also that

$$ns_n + (s_1 + s_2 + s_3 + \&c., \text{ to } n \text{ terms}) \\ = \frac{s_1}{r-1} \left\{ n(r^n - 2) + \frac{r}{r-1} (r^n - 1) \right\}.$$

If a, b, c, d are in geometrical progression, show that:

29. $(a^2 + b^2)(c^2 + d^2) = b^2(b + d)^2$.

30. $a^3 + b^3 + c^3 = (a + b + c)(a - b + c)$.

31. $3(a - b + c)^2 - (a^3 + b^3 + c^3) = 2(a - b + c)(a - 2b + c)$.

32. $(a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(b + c)^2$.

33. $a^2(a - 2b + c)^2 = (a - b)^4$.

34. $a^4(a - 3b + 3c - d)^2 = (a - b)^6$.

35. Find the sum of n terms of the series whose n th term is $2^n + n$.

36. If a, b, c, d are in geometrical progression, show that $(a - b)^2, (b - c)^2, (c - d)^2$ are in G. P.

Harmonical Progression.

131. Quantities are said to be in **Harmonical Progression** when any three consecutive terms being taken, the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus, if a, b, c be in H. P., we must have

$$a : c :: a - b : b - c.$$

We cannot make a formula for the sum of n terms of an H. P., as in the cases of Arithmetical and Geometrical Progression.

132. *The reciprocals of the terms of an Harmonical Progression form a series in Arithmetical Progression.*

Let a, b, c be in H. P.

Then Art. (131), $a : c :: a - b : b - c$,

$\therefore a(b - c) = c(a - b)$,
or dividing each side by abc ,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

that is, the difference between the first and second of the terms $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ is equal to the difference between the second and third.

Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

133. To insert m harmonic means between a and b .

Taking the reciprocals $\frac{1}{a}$ and $\frac{1}{b}$, we must first find m arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and afterwards invert them.

Let d be the common difference of the arithmetical series.

$$\text{Then (Art. 126), } d = \frac{\frac{1}{b} - \frac{1}{a}}{m + 1} = \frac{a - b}{(m + 1)ab}.$$

The arithmetical means are

$$\frac{1}{a} + \frac{a - b}{(m + 1)ab}, \frac{1}{a} + \frac{2(a - b)}{(m + 1)ab}, \&c., \frac{1}{a} + \frac{m(a - b)}{(m + 1)ab}, \text{ or } \\ \frac{a + mb}{(m + 1)ab}, \frac{2a + m - 1 \cdot b}{(m + 1)ab}, \&c., \frac{ma + b}{(m + 1)ab}.$$

Hence, inverting, the harmonic means are

$$\frac{(m + 1)ab}{a + mb}, \frac{(m + 1)ab}{2a + m - 1 \cdot b}, \&c., \frac{(m + 1)ab}{ma + b}.$$

134. If A, G, H respectively represent the Arithmetical, Geometrical, and Harmonical means between a and c , show that $A : G :: G : H$.

We have $A - a = c - A$, and $\therefore A = \frac{1}{2}(a + c) \dots\dots (1)$.

Again, $\frac{G}{a} = \frac{c}{G}$, and $\therefore G = \sqrt{ac} \dots\dots\dots (2)$.

Also, $a : c :: H - a : c - H$ and $\therefore H = \frac{2ac}{a + c} \dots\dots\dots (3)$.

Multiplying (1) and (3), we have

$$AH = \frac{1}{2}(a + c) \cdot \frac{2ac}{a + c} = ac \dots \dots \dots (4).$$

And from (2) we get $G^2 = ac \dots \dots \dots (5).$

(4) = (5), then $AH = G^2$, or $A : G :: G : H$.

COR. $A > G$, and $G > H$.

$$\begin{aligned} \text{For, (1) - (2), } A - G &= \frac{1}{2}(a + c) - \sqrt{ac} \\ &= \frac{1}{2}(a - 2\sqrt{ac} + c) \\ &= \frac{1}{2}(\sqrt{a} - \sqrt{c})^2, \text{ a positive quantity.} \\ \therefore A &> G. \end{aligned}$$

And hence, since A, G, H form a continued proportion, it must follow that $G > H$. *Q.E.D.*

Ex. 1. The 6th term of an H. P. is $\frac{1}{13}$, and the 1st term is $\frac{1}{3}$, find the intermediate terms.

Inverting, we have 3 and 13 as the 1st and 6th terms of the corresponding A. P.

Hence $13 = 3 + 5d$, or $d = 2$.

Therefore the intermediate terms of the A. P. are 5, 7, 9, 11.

Hence the required terms are $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$.

Ex. 2. If the first three of four numbers are in A. P., and the last three in H.P., then the product of the means equals the product of the extremes.

Let a, b, c, d be the four numbers.

Then, since a, b, c are in A. P., we have, by Art. 134,

$$b = \frac{1}{2}(a + c) \dots \dots \dots (1).$$

And, since b, c, d are in H. P., we have also, by Art. 134,

$$c = \frac{2bd}{b + d} \dots \dots \dots (2).$$

Substituting in (2) for the value of b in (1), we have

$$c = \frac{(a + c)d}{b + d}, \text{ or } bc + cd = ad + cd;$$

$$\therefore bc = ad. \quad Q.E.D.$$

. Ex. XXXII.

1. Insert 7 harmonic means between 1 and $\frac{1}{4}$.
2. Insert 6 harmonic means between 2 and $\frac{3}{4}$.
3. Continue to 4 terms each way the series 2, $1\frac{1}{2}$, $1\frac{1}{3}$.
4. If P, Q, R are respectively the p th, q th, and r th terms of an H. P., show that

$$QR(q - r) + RP(r - p) + PQ(p - q) = 0.$$

5. Given P, Q, the p th and q th terms respectively, find the n th term.

6. What quantity must be added to each of the quantities a, b, c in order to form an H. P.?

7. If s, s' be the sums of two infinite series whose first term is 1 and common ratios r and r' , then will $s, s', \frac{1}{r+1-2r'}$, be in H. P.

8. If a, b, c are in H. P., show that

$$a - b : a :: a - c : a + c.$$

9. If G, H be the geometric and harmonic means between a and b , show that

$$b = G \cdot \frac{\sqrt{G+H} - \sqrt{G-H}}{\sqrt{G+H} + \sqrt{G-H}}.$$

10. The sum of three numbers in H. P. is $1\frac{1}{2}$, and the sum of their reciprocals is $8\frac{1}{2}$. Find the numbers.

11. If H be the harmonic mean between a and b , and H_1 the harmonic mean between a and H, and also H_2 the harmonic mean between H and b , show that

$$H_1 : H_2 :: a + 3b : 3a + b.$$

12. If A_1, A_2 , be the arithmetic means corresponding to the harmonic means H_1, H_2 in the last example, show that

$$A_1 H_1 = A_2 H_2.$$

CHAPTER XVI.

THE BINOMIAL THEOREM.

135. We have already shown (Art. 32) how to develop by multiplication the positive integral powers of $a + b$, and indeed by *assuming* the truth of an apparent general law, shown how to develop these powers by inspection. We shall now prove this law, which goes by the name of the **Binomial Theorem**, and may be expressed as follows:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \&c.$$

where n may be integral or fractional, positive or negative.

Positive Integral Index.

136. To prove the *Binomial Theorem* for a positive integral index.

By actual multiplication, we have

$$\begin{aligned} (a + b_1)(a + b_2) &= a^2 + (b_1 + b_2)a + b_1b_2, \\ (a + b_1)(a + b_2)(a + b_3) &= a^3 + (b_1 + b_2 + b_3)a^2 + (b_1b_2 + b_1b_3 + b_2b_3)a + b_1b_2b_3, \\ (a + b_1)(a + b_2)(a + b_3)(a + b_4) &= a^4 + (b_1 + b_2 + b_3 + b_4)a^3 + (b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)a + b_1b_2b_3b_4. \end{aligned}$$

We observe—as on Page 48, (1), (2)—the following laws:

1. The coefficient of the *first* term is unity; that of the *second* term the sum of the letters b_1, b_2, b_3 , &c., taken *singly*; that of the *third*, the sum of their products taken *two* together; that of the *fourth*, the sum of their products taken *three* together; and so on, the last term being the product of all the letters b_1, b_2, b_3 , &c.

2. The index of a is, in the first term, that of the number of binomial factors, and it *sinks one* every term.

3. The number of terms is *one more* than the number of binomial factors.

We will assume this law to be true for r factors. We then have

$$(a + b_1)(a + b_2) \dots (a + b_r) = a^r + p_1 a^{r-1} + p_2 a^{r-2} + p_3 a^{r-3} + \&c. + p_r \dots (1),$$

Where p_1 = the sum of b_1, b_2, b_3 , &c., b_r taken *singly*,

p_2 = " " the products taken *two* together,

p_3 = " " *three* "

&c. = &c.

p_r = the product of all the letters b_1, b_2, b_3 , &c., b_r

Multiply each side of (1) by $a + b_{r+1}$, then

$$(a + b_1)(a + b_2)(a + b_3) \dots (a + b_r)(a + b_{r+1}) \\ = a^{r+1} + (p_1 + b_{r+1}) a^r + (p_2 + p_1 b_{r+1}) a^{r-2} + (p_3 + p_2 b_{r+1}) a^{r-1} + \&c. + p_r b_{r+1} \dots (2).$$

Now on the first side of (2) we have the product of $(r + 1)$ binomial factors.

And on the second side,

$$p_1 + b_{r+1} = b_1 + b_2 + b_3 + \&c. + b_r + b_{r+1} \\ = \text{the sum of the letters } b_1, b_2, b_3, \&c., b_r + b_{r+1} \text{ taken singly;}$$

$p_2 + p_1 b_{r+1}$ = the sum of the products of the letters b_1, b_2, b_3 &c., b_{r+1} taken *two* together.
 $p_3 + p_2 b_{r+1}$ = " " " " " " " " *three* together.
 &c. = &c.

$p_1 b_{r+1}$ = product of all the $(r + 1)$ letters b_1, b_2, b_3 &c., b_{r+1} .

Hence in (2) we have shown that by assuming the above law to hold for r binomial factors, it also holds for *one factor more*. Now we know by actual multiplication that it holds for 2, 3, 4 binomial factors.

It therefore holds for 5, and therefore also for 6, 7, &c. Hence it is generally true for any number of binomial factors.

Let us now suppose each of the quantities b_1, b_2, b_3 &c., to be equal to b , and let there be n binomial factors $a + b$.

Then we have for their product $(a + b)(a + b)(a + b) \dots$ to n factors or $(a + b)^n$.

Hence

$$(a + b)^n = a^n + p_1 a^{n-1} + p_2 a^{n-2} + \&c. + p_n \dots \dots \dots (3).$$

Now $p_1 = b + b + b + \&c.$ to n terms $= nb$.

$p_2 = bb + bb + bb + \&c.$ to as many terms as there are combinations of n things taken two together, and therefore, by Art. 128, to $\frac{n(n-1)}{1 \cdot 2}$ terms.

$$\therefore p_2 = \frac{n(n-1)}{1 \cdot 2} bb = \frac{n(n-1)}{1 \cdot 2} b^2.$$

$$\text{And so } p_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} b^3 \&c.$$

Also $p_n = b \cdot b \cdot b \dots$ to n factors $= b^n$.

Hence, substituting in (3), we have $(a + b)^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 \\ + \&c. + b^n. \quad Q.E.D.$$

For exercises upon this formula, the student is referred to Ex. X.

COR. 1. If $a = 1, b = x$, we have, since $a^n, a^{n-1}, \&c.$, each $= 1$,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 \\ + \&c. + x^n. \\ = 1 + C_1x + C_2x^2 + C_3x^3 + \&c. + C_nx^n.$$

where $C_1, C_2, C_3, \&c., C_n$ represent respectively the number of combinations of n things taken 1, 2, 3, &c., n together.

COR. 2. Since the coefficient of the $(r+1)$ th term is C_r , we have, by Art. 128,

The $(r+1)$ th term of the expansion of $(1 + x)^n$

$$= \frac{n(n-1)(n-2) \dots (n-r+1)}{\underbrace{1 \cdot 2 \cdot 3 \dots r}} x^r, \text{ or, as it may be written,} \\ = \frac{\underbrace{n}_{\downarrow}}{\underbrace{r}_{\downarrow} \underbrace{n-r}_{\downarrow}} x^r.$$

137. *The coefficients of the terms equidistant from the beginning and end are equal.*

For (Art. 136, Cor. 1) the coefficient of the $(r+1)$ th term from the beginning is C_r , and that of the $(r+1)$ th term from the end is C_{n-r} .

Now, in Art. 129, we have shown that $C_r = C_{n-r}$, and hence the truth of the proposition.

138. *To find the middle term of the expansion of $(1 + x)^n$.*

(1.) Let n be even.

Then, as there are $(n + 1)$ terms in the expansion, the middle term will be the $\left(\frac{n}{2} + 1\right)$ th term.

Hence (Art. 136, Cor. 2) putting $\frac{n}{2}$ for r , we have

Middle term

$$\begin{aligned}
 &= \frac{n(n-1)(n-2)\dots(n-\frac{n}{2}+1)}{1.2.3\dots\frac{n}{2}} x^{\frac{n}{2}}. \\
 &= \frac{n(n-1)(n-2)\dots(\frac{n}{2}+1)}{1.2.3\dots\frac{n}{2}} x^{\frac{n}{2}}, \dots\dots\dots (1).
 \end{aligned}$$

We may transform this expression thus :

Middle term

$$\begin{aligned}
 &= \frac{n(n-1)(n-2)\dots(\frac{n}{2}+1) \cdot \frac{n}{2}(\frac{n}{2}-1)\dots 2 \cdot 1}{1.2.3\dots\frac{n}{2} \cdot 1.2\dots\frac{n}{2}} x^{\frac{n}{2}} \\
 &= \frac{n(n-1)(n-2)\dots 4 \cdot 3 \cdot 2 \cdot 1}{(1.2.3\dots\frac{n}{2})^2} x^{\frac{n}{2}};
 \end{aligned}$$

or, separating the even and odd factors,

$$\begin{aligned}
 &= \frac{\{n(n-2)\dots 4 \cdot 2\} \times \{(\frac{n}{2}-1)\dots 3 \cdot 1\}}{(1.2.3\dots\frac{n}{2})^2} x^{\frac{n}{2}} \\
 &= \frac{2^{\frac{n}{2}} \left\{ \frac{n}{2}(\frac{n}{2}-1)\dots 2 \cdot 1 \right\} \cdot \{1 \cdot 3 \cdot 5 \dots (n-1)\}}{(1.2.3\dots\frac{n}{2})^2}
 \end{aligned}$$

or, striking out common factors from numerator and denominator,

$$= 2^{\frac{n}{2}} \frac{1 \cdot 3 \cdot 5 \dots (n-1)}{1 \cdot 2 \cdot 3 \dots \frac{n}{2}} x^{\frac{n}{2}}.$$

(2.) Let n be odd.

As there are $(n+1)$ or an even number of terms in the expansion, there will evidently be two middle terms. These are the $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+1}{2} + 1\right)$ th terms.

The coefficient of each may be easily shown to be

$$\frac{n(n-1)(n-2)\dots\frac{n+1}{2}}{1 \cdot 2 \cdot 3 \dots \frac{n+1}{2}}, \text{ or } 2^{\frac{n-1}{2}} \cdot \frac{1 \cdot 3 \cdot 5 \dots n}{1 \cdot 2 \cdot 3 \dots \frac{n+1}{2}}.$$

139. To find the greatest coefficient in the expansion of $(1+x)^n$, n being an integer.

Since (Art. 136, Cor. 1) C_r represents the coefficient of the $(r+1)$ th term, and we have shown that C_r is greatest, if n be even, when $r = \frac{n}{2}$; and if n be odd, when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ (see Art. 123), it follows that:

(1.) When n is even, the greatest coefficient is the $\left(\frac{n}{2} + 1\right)$ th;

(2.) When n is odd, the greatest coefficients are the $\left(\frac{n-1}{2} + 1\right)$ th and $\left(\frac{n+1}{2} + 1\right)$ th, or the $\left(\frac{n+1}{2}\right)$ th and the $\left(\frac{n+3}{2}\right)$ th coefficients.

These evidently correspond to the coefficients of the middle terms.

The Binomial Theorem for any Index.

140. Let $f(m)$ represent the algebraical expression

$$1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

where m may be either integral or fractional, positive or negative.

Hence we have

$$f(m) = 1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \&c. \dots \dots \dots (1),$$

and, putting n for m ,

$$f(n) = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \&c. \dots \dots \dots (2).$$

Now, multiplying together (1) and (2) and arranging, we have

$$f(m) f(n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2} x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

But, putting $(m+n)$ for m in (1), the expression on the right side of this equation is equal to $f(m+n)$.

Hence we have

$$f(m) \cdot f(n) = f(m+n) \dots \dots \dots (3).$$

Hence, also,

$$f(m) \cdot f(n) \cdot f(p) = f(m+n) \cdot f(p), \text{ or, by (3), } = f(m+n+p).$$

And generally, $f(m) \cdot f(n) \cdot f(p) \dots f(k) = f(m + n + p + \dots + k) \dots \dots \dots (4)$,
 when $m, n, p, \dots k$ may be either positive or negative, integral or fractional.

Let $m = n = p = \&c. = k = \frac{r}{s}$, where r and s are positive integers.

and suppose there are s of the quantities $m, n, p, \&c., k$.

Then $m + n + p + \&c. + k = s$ times $\frac{r}{s} = r$,

And $f(m) \cdot f(n) \cdot f(p) \dots f(k) = \left\{ f\left(\frac{r}{s}\right) \right\}^s$.

Hence (4) becomes

$$\left\{ f\left(\frac{r}{s}\right) \right\}^s = f(r), \text{ or, taking the } s\text{th root of each side,}$$

$$f\left(\frac{r}{s}\right) = \left\{ f(r) \right\}^{\frac{1}{s}} \dots \dots \dots (5).$$

But, by (i),

$$f(r) = 1 + rx + \frac{r(r-1)}{1 \cdot 2} x^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

And hence, by Art. 35, since r is a positive integer,

$$f(r) = (1 + x)^r$$

We have therefore

$$\{f(r)\}^{\frac{1}{s}} = \{(1+x)r\}^{\frac{1}{s}} = (1+x)^{\frac{r}{s}}.$$

And therefore from (5)

$$f\left(\frac{r}{s}\right) = (1+x)^{\frac{r}{s}} \dots\dots\dots (6).$$

But from (1), putting $\frac{r}{s}$ for m ,

$$f\left(\frac{r}{s}\right) = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1 \cdot 2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1 \cdot 2 \cdot 3}x^3 + \&c. \dots\dots\dots (7).$$

Hence, equating (6) and (7),

$$(1+x)^{\frac{r}{s}} = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1 \cdot 2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1 \cdot 2 \cdot 3}x^3 + \&c. \dots\dots\dots (A).$$

which proves the Binomial Theorem for a positive fractional index.

It is therefore true for a positive index, whether integral or fractional.

We shall now show it is true for a negative index.

We have in (3),

$$f(m) \cdot f(n) = f(m+n),$$

when there is no restriction as to the value of m and n .

Let $n = -m$, when m is a positive quantity.

We then have $f(m) \cdot f(-m) = f(m-m) = f(0).$

But from (2), putting $n = 0$, we have $f(0) = 1$, and hence

$$f(m) \cdot (-m) = 1$$

$$\therefore f(-m) = \frac{1}{f(m)} \dots\dots\dots (8).$$

Now we have just shown that whether m be integral or fractional, so that it be positive, $f(m) = (1+x)^m$

And therefore
$$\frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1+x)^{-m}.$$

We have therefore from (8)

$$f(-m) = (1+x)^{-m} \dots\dots\dots (9).$$

But from (2), putting $-m$ for n , we have

$$f(-m) = 1 + (-m)x + \frac{-m(-m-1)}{1 \cdot 2} x^2 + \frac{-m(-m-1)(-m-2)}{1 \cdot 2 \cdot 3} x^3 + \&c. \dots (10).$$

Hence, equating (9) and (10), we have

$$(1+x)^{-m} = 1 + (-m)x + \frac{-m(-m-1)}{1 \cdot 2} x^2 + \frac{-m(-m-1)(-m-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

which proves the Binomial Theorem for any negative index.

We have therefore shown it to be true for any index.

141. To find the sum of the coefficients of the expansion of $(1 + x)^n$.

Allowing $C_1, C_2, C_3, \&c.$, to represent the coefficients of the first, second, third, &c., powers of x , we have

$$(1 + x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \&c. \dots \dots \dots (1).$$

Put $x = 1$; then, since the powers of x are each $= 1$, we have

$$2^n = 1 + C_1 + C_2 + C_3 + C_4 + \&c.$$

$$\therefore 1 + C_1 + C_2 + C_3 + C_4 + \&c. = 2^n.$$

COR. 1. Since, when n is a positive integer, $C_1, C_2, C_3, \&c.$, represent the number of combinations of n things when taken 1, 2, 3, &c., together respectively, it follows that
The total number of combinations of n things $= 2^n - 1$.

COR. 2. Putting $x = -1$, we have

$$(1 - 1)^n \text{ or } 0 = 1 - C_1 + C_2 - C_3 + \&c., \text{ or}$$

$$C_1 + C_3 + C_5 + \&c. = 1 + (C_2 + C_4 + C_6 + \&c.)$$

Hence the number of odd groups which can be made of n things exceeds by unity the number of even groups.

142. To find the GREATEST TERM in the expansion of $(1 + x)^n$, no regard being had to the sign of the term.

$$\text{By Art. 136, Cor. 2, } (r + 1)\text{th term} = \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} x^r.$$

$$= \frac{n(n-1)(n-2) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots r-1} x^{r-1} \cdot \frac{n-r+1}{r} x \dots (1).$$

$$\text{And } r\text{th term} = \frac{n(n-1)(n-2) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots r-1} x^{r-1} \dots (2).$$

Hence, (1) $\div$ (2), we have

$$\frac{(r+1)\text{th term}}{r\text{th term}} = \frac{n-r+1}{r} x = \left(\frac{n+1}{r} - 1 \right) x \dots (3).$$

We may therefore obtain the $(r+1)$ th term from the r th by multiplying the latter by $\left(\frac{n+1}{r} - 1 \right) x$.

And as long as $\left(\frac{n+1}{r} - 1 \right) x$ continues to be greater than unity, the $(r+1)$ th term is greater than the r th term.

Now, as r is increased, $\left(\frac{n+1}{r} - 1 \right) x$ diminishes, and eventually changes its sign.

And if r be indefinitely increased, we may make the value of $\left(\frac{n+1}{r} - 1 \right) x$ approach as near to $0 - x$ as we please.

Hence, if none of the values of $\left(\frac{n+1}{r} - 1 \right) x$ is zero, we may, by taking r large enough, make any term as nearly equal to x times the preceding term as we please. (It must be remembered that we are not considering here the sign of the term).

Now $\left(\frac{n+1}{r} - 1\right)x = 0$, when $r = n + 1$, and this cannot be except when n is integral and positive. From this we learn that there are only $(n + 1)$ terms in the expansion of $(1 + x)^n$ when n is an integer and positive, and that there is no limit to the number of terms when n is negative or fractional.

It follows therefore that *when n is fractional or negative, there cannot possibly be a greatest term if x be > 1* , since by increasing r sufficiently any term may approach as nearly to x times the preceding as we please, and there is no limit to the value of r .

We will now consider separately the cases in which a greatest term is possible.

It will be necessary only to consider x as positive; for when x is negative, we may put $x = -y$, and find the greatest term in the expansion of $(1 + y)^n$, and this will be the greatest term required.

I. *Let n be integral and positive.*

Now when $\left(\frac{n+1}{r} - 1\right)x = 1$, we have $r = \frac{(n+1)x}{x+1}$.

(a.) *Let $\frac{(n+1)x}{x+1}$ be integral.*

Then there is a term corresponding to the value of this fraction such that it and the next succeeding term have the same magnitude, and are the greatest terms of the expansion.

Hence the r th and $(r + 1)$ th terms are the greatest terms when $r = \frac{(n+1)x}{x+1}$ an integer.

(b.) *Let $\frac{(n+1)x}{x+1}$ be a fraction.*

Then $\left(\frac{n+1}{r} - 1\right)x < 1$, when the value of r is the in-

teger next greater than $\frac{(n+1)x}{x+1}$.

Hence the r th term is the greatest term for the value of r corresponding to the integer next greater than $\frac{(n+1)x}{x+1}$.

II. *Let n be fractional and positive.*

It has been shown that *when x is greater than unity there cannot be a greatest term.*

When x is less than unity the conditions are exactly as in I. above.

III. *Let n be negative.*

We have shown that here, as in II., *there is no greatest term when x is greater than unity.*

When x is less than unity, let $n = -m$.

$$\text{Then } \left(\frac{n+1}{r} - 1 \right) x = - \left(\frac{m-1}{r} + 1 \right) x.$$

$$\text{When } \left(\frac{m-1}{r} + 1 \right) x = 1, \text{ we have } r = \frac{(m-1)x}{1-x}.$$

(a.) *Let $\frac{(m-1)x}{1-x}$ be integral and positive.*

Then, reasoning as before, the r th and $(r+1)$ th are the greatest terms when $r = \frac{(m-1)x}{1-x}$ a positive integer.

(b.) *Let $\frac{(m-1)x}{1-x}$ be fractional and positive.*

Then, as above, the r th is the greatest term for that value of r corresponding to the integer next greater than $\frac{(m-1)x}{1-x}$.

(c.) *Let $\frac{(m-1)x}{1-x}$ be negative.*

In this case, since x is less than unity, m is also less than unity.

Hence $\left(\frac{m-1}{r} + 1\right)x$ is always less than x , and therefore less than unity for every value of r .

It follows therefore that when $\frac{(m-1)x}{1-x}$ is negative—that is, when x is less than unity, and m is less than unity—the first term of the expansion is the greatest.

Cor. 1. Since, when $\frac{(n+1)x}{x+1}$ is < 1 , we have $nx < 1$,

And when $\frac{(m-1)x}{1-x}$ is < 1 , x being less than unity, we have $mx < 1$,

It follows that the conditions that the first shall not be the greatest term are :

- (1) When n is a positive integer, nx must be > 1 .
- (2) When n is fractional or negative.
 - (a.) If x be > 1 , this condition is sufficient.
 - (b.) If x be < 1 , nx must be > 1 .

Cor. 2. Hence, also, in order that the last term shall not be the greatest term, since we need only consider the case of n being a positive integer, the condition is that $\frac{n}{x}$ is > 1 , or $n > x$.

Approximate Values.

143. To find the n th root of N , when N is very nearly an n th power of a rational quantity.

Let $N = a^n + b$, where b is small compared with a^n (1).

$$\begin{aligned}
\text{Then } N_n^{\frac{1}{2}} &= (a^n + b)^{\frac{1}{2}} = a \left(1 + \frac{b}{a^n} \right)^{\frac{1}{2}} \\
&= a \left\{ 1 + \frac{1}{n} \cdot \frac{b}{a^n} + \frac{1}{1 \cdot 2} \cdot \frac{1}{n} \left(\frac{1}{n} - 1 \right) \cdot \left(\frac{b}{a^n} \right)^2 + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{n} \left(\frac{1}{n} - 1 \right) \left(\frac{1}{n} - 2 \right) \cdot \left(\frac{b}{a^n} \right)^3 + \&c. \right\} \\
&= a \left\{ 1 + \frac{1}{n} \cdot \frac{b}{a^n} - \frac{n-1}{1 \cdot 2 \cdot n^2} \cdot \frac{b^2}{a^{2n}} + \frac{(n-1)(2n-1)}{1 \cdot 2 \cdot 3 \cdot n^3} \cdot \frac{b^3}{a^{3n}} - \&c. \right\}
\end{aligned}$$

Now, as b is small compared with a^n , the values of the successive terms of this expansion diminish rapidly, and hence we may approximate as nearly as we please to the value of $N_n^{\frac{1}{2}}$.

Ex. 1. Find the 7th root of 2192.

We have $N = 2192 = 2187 + 5 = 3^7 + 5 = 3^7 \left(1 + \frac{5}{3^7} \right)$.

$\therefore N^{\frac{1}{7}} = 3 \left(1 + \frac{5}{3^7} \right)^{\frac{1}{7}}$, from which by expansion, &c., we get

$$\sqrt[7]{2192} = 3 \left\{ 1 + \frac{1}{7} \cdot \frac{5}{3^7} - \frac{6}{1 \cdot 2 \cdot 7^2} \cdot \left(\frac{5}{3^7} \right)^2 + \frac{6 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 7^3} \cdot \left(\frac{5}{3^7} \right)^3 - \&c. \right\}.$$

Ex. 2. Find the 5th root of 1021.

Here $N = 1021 = 1024 - 3 = 4^5 - 3 = 4^5 \left(1 - \frac{3}{4^5} \right)$.

$$\therefore N^{\frac{1}{2}} = 4 \left(1 - \frac{3}{4^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$= 4 \left\{ 1 - \frac{1}{5} \cdot \frac{3}{4^{\frac{1}{2}}} - \frac{4}{1 \cdot 2 \cdot 5^{\frac{1}{2}}} \cdot \left(\frac{3}{4^{\frac{1}{2}}}\right)^2 - \frac{4 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 5^{\frac{3}{2}}} \cdot \left(\frac{3}{4^{\frac{1}{2}}}\right)^3 - \&c. \right\}$$

144. If c is large compared with a or b , show that

$$(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}}$$

$$= 2c^{-(m+1)} - (m+1)c^{-(m+1)} \left\{ a^2 - (m+2)b^2 \right\}^{-\frac{m+1}{2}} \text{ nearly.}$$

We have

$$= c^{-(m+1)} \left\{ 1 - \left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2} \right) \right\}^{-\frac{m+1}{2}}; \text{ expanding, \&c., then}$$

$$= c^{-(m+1)} \left\{ 1 + \frac{m+1}{2} \left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2} \right) + \frac{(m+1)(m+3)}{8} \left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2} \right)^2 + \&c. \right\}$$

$$= c^{-(m+1)} \left\{ 1 + \frac{m+1}{2} \left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2} \right) + \frac{(m+1)(m+3)}{8} \cdot \frac{4b^2}{c^2} \right\} \dots\dots\dots (1),$$

where we retain no terms involving higher powers of c than the second.

And similarly

$$(a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}} = c^{-(m+1)} \left\{ 1 - \frac{m+1}{2} \left(\frac{2b}{c} + \frac{a^2 + b^2}{c^2} \right) + \frac{(m+1)(m+3)}{8} \cdot \frac{4b^2}{c^2} \right\} \dots\dots\dots (2).$$

(1) + (2), then

$$\begin{aligned} & (a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}} \\ &= c^{-(m+1)} \left\{ 2 - (m+1) \frac{a^2 + b^2}{c^2} + (m+1)(m+3) \cdot \frac{b^2}{c^2} \right\} \\ &= 2c^{-(m+1)} - (m+1)c^{-(m+2)} \{ a^2 - (m+2)b^2 \}. \quad Q.E.D. \end{aligned}$$

This is an important result in the mathematical theory of magnetism.

145. When $n + 1$ figures of a square root have been obtained by the ordinary method, n figures more may be found by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is $2n + 1$.

Let N be the number whose square root is required, a the part already obtained, and b the part required.

Then $\sqrt{N} = a + b$

$$\therefore N = a^2 + 2ab + b^2 \text{ or } N - a^2 = 2ab + b^2.$$

$$\therefore \frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

Now $N - a^2$ is the remainder after obtaining the part a of the square root by the ordinary method.

And we see that $N - a^2$ when divided by $2a$, gives the remaining part b of the square root, together with the quantity $\frac{b^2}{2a}$.

We have then to show that $\frac{b^3}{2a}$ is a proper fraction, and therefore has no influence upon the integral value of the remaining figures of the root.

We assume that b contains n digits; and therefore b^3 contains not more than $2n$ digits.

But a , since it contains the $n + 1$ figures of the root already found, and also n ciphers, occupying the place of those already to be found, must contain $2n + 1$ digits.

Hence $\frac{b^3}{2a}$ is less than unity, and the proposition is proved.

This result has been assumed and illustrated by an example on p. 56.

146. *When $n + 2$ figures of a cube root have been obtained by the ordinary method, n more figures may be found by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is $2n + 2$.*

Let N be the number whose cube root is required, a the part already obtained, and b the part required.

$$\begin{aligned}\text{Then} \quad \sqrt[3]{N} &= a + b \\ \therefore N &= a^3 + 3a^2b + 3ab^2 + b^3, \\ \text{or} \quad N - a^3 &= 3a^2b + 3ab^2 + b^3 \\ \therefore \frac{N - a^3}{3a^2} &= b + \frac{b^2}{a} + \frac{b^3}{3a^2}.\end{aligned}$$

Now $N - a^3$ represents the remainder after obtaining the part a of the cube root, and $3a^2$ is the next trial divisor.

Hence we see that on dividing the remainder by the next trial divisor we get the required quotient b , together with the quantity $\frac{b^2}{a} + \frac{b^3}{3a^2}$.

Now, in order that this may not affect the integral value of the remaining figures of the cube root, we must show that $\frac{b^2}{a} + \frac{b^3}{3a^2}$ is less than unity.

Since b contains n figures, we must have $b < 10^n$.

And since a contains $2n + 2$ figures, a cannot be less than 10^{2n+1} , but may be equal to or greater than 10^{2n+1} .

$$\begin{aligned} \therefore \frac{b^2}{a} + \frac{b^2}{3a^2} \text{ is } &< \frac{(10^n)^2}{10^{2n+1}} + \frac{(10^n)^2}{3(10^{2n+1})^2} \\ &< \frac{1}{10} + \frac{1}{3 \cdot 10^{n+2}}, \text{ and } \therefore < 1. \end{aligned}$$

Hence the truth of the proposition.

This result has been assumed and illustrated in p. 62.

147. We shall now work out a few examples on the Binomial Theorem.

Ex. 1. Expand to four terms the expression $(1 + 3x)^{-\frac{5}{3}}$.

$$\begin{aligned} (1 + 3x)^{-\frac{5}{3}} &= 1 + \left(-\frac{5}{3}\right) \cdot 3x + \frac{-\frac{5}{3} \left(-\frac{5}{3} - 1\right)}{1 \cdot 2} (3x)^2 \\ &\quad + \frac{-\frac{5}{3} \left(-\frac{5}{3} - 1\right) \left(-\frac{5}{3} - 2\right)}{1 \cdot 2 \cdot 3} (3x)^3 + \&c. \\ &= 1 - 5x + 20x^2 - \frac{220}{3}x^3 + \&c. \end{aligned}$$

Ex. 2. Find the $(r + 1)$ th term of the expansion of $(2 - 5x)^{-4}$.

$(r + 1)$ th term

$$\begin{aligned} &= \frac{-4(-4-1)(-4-2)\dots(-4-r+1)}{1 \cdot 2 \cdot 3 \dots r} 2^{-4-r} (-5x)^r \\ &= \frac{(-1)^r \cdot 4 \cdot 5 \cdot 6 \dots (r+3)}{1 \cdot 2 \cdot 3 \dots r} 2^{-4-r} (-1)^r (5x)^r \\ &= (-1)^{2r} \cdot \frac{4 \cdot 5 \cdot 6 \dots r(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \dots r} 2^{-4-r} (5x)^r; \\ &\quad \text{or, since } (-1)^{2r} = 1, \\ &= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3} \cdot \frac{(5x)^r}{2^{r+4}}. \end{aligned}$$

Ex. 3. If x be very small, show that

$$\frac{(1 + 2x)^{\frac{1}{2}} + (1 + 3x)^{\frac{1}{2}}}{2 + 5x - (1 + 4x)^{\frac{1}{2}}} = 2 - 4x, \text{ very nearly.}$$

Expanding and retaining terms containing the *first powers* of x only, we have

$$\begin{aligned} & \frac{(1 + 2x)^{\frac{1}{2}} + (1 + 3x)^{\frac{1}{2}}}{2 + 5x - (1 + 4x)^{\frac{1}{2}}} \\ &= \frac{(1 + \frac{1}{2} \cdot 2x) + (1 + \frac{1}{2} \cdot 3x)}{2 + 5x - (1 + \frac{1}{2} \cdot 4x)} \\ &= \frac{2 + 2x}{1 + 3x} = (2 + 2x) \cdot (1 + 3x)^{-1}; \end{aligned}$$

or, expanding as before,

$$= (2 + 2x) (1 - 3x);$$

or multiplying out, and neglecting the term involving x^2 ,

$$= 2 - 4x. \quad Q.E.D.$$

Ex. 4. If p and q be very nearly equal, and n an integer, show that

$$\left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} \text{ nearly.}$$

$$\begin{aligned} \text{We have } \frac{p}{q} &= \frac{p+q}{\frac{2q}{p+q}} = \frac{p+q}{\frac{(p+q) - (p-q)}{p+q}} \\ &= \frac{1 + \frac{p-q}{p+q}}{1 - \frac{p-q}{p+q}} \end{aligned}$$

$$\therefore \left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{\left(1 + \frac{p-q}{p+q}\right)^{\frac{1}{n}}}{\left(1 - \frac{p-q}{p+q}\right)^{\frac{1}{n}}};$$

expanding and retaining only two terms, since $p - q$ is very small,

$$\begin{aligned} & 1 + \frac{1}{n} \cdot \frac{p-q}{p+q} \\ &= \frac{1 + \frac{1}{n} \cdot \frac{p-q}{p+q}}{1 - \frac{1}{n} \cdot \frac{p-q}{p+q}} = \frac{n(p+q) + (p-q)}{n(p+q) - (p-q)} \\ &= \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} \quad Q.E.D. \end{aligned}$$

Ex. 5. Find the limit of $\left(1 + \frac{1}{x}\right)^x$, when x increases without limit.

Expanding by the Binomial Theorem, we have

$$\begin{aligned}\left(1 + \frac{1}{x}\right)^x &= 1 + \frac{x}{1} \cdot \frac{1}{x} + \frac{x(x-1)}{1 \cdot 2} \cdot \frac{1}{x^2} + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{x^3} + \&c. \\ &= 1 + \frac{1}{1} + \frac{1 - \frac{1}{x}}{1 \cdot 2} + \frac{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)}{1 \cdot 2 \cdot 3} + \&c.\end{aligned}$$

When x increases without limit, each of the quantities $\frac{1}{x}, \frac{2}{x}, \frac{3}{x}, \&c.$, decreases without limit, and becomes ultimately zero.

Hence, when x is infinite,

$$\begin{aligned}\left(1 + \frac{1}{x}\right)^x &= 1 + \frac{1}{1} + \frac{1 - 0}{1 \cdot 2} + \frac{(1 - 0)(1 - 0)}{1 \cdot 2 \cdot 3} + \&c. \\ &= 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \&c.\end{aligned}$$

It is usual to denote this series by the symbol e ; it is the base of the Napierian system of logarithms.

We therefore have

$$\text{Limit of } \left(1 + \frac{1}{x}\right)^x \text{ when } x \text{ is infinite} = e.$$

The approximate value of e is 2.71828.

Ex. XXXIII.

Expand to four terms the six following expressions:

1. $(1 + x)^7$.
2. $(3 + 2x)^{\frac{1}{2}}$.
3. $(4 - 5x)^{-\frac{2}{3}}$.
4. $(a + bx^2)^{-1}$.
5. $(1 - 2x + 3x^2)^{-2}$.
6. $\left(\frac{a}{x} - \frac{b}{y}\right)^{-5}$.

Find the middle terms of:

7. $(a + b)^9$.
8. $(3 + 5x)^{10}$.
9. $(a^2 - b^2)^6$.
10. $\left(x + \frac{1}{x}\right)^{2n}$.
11. $\left(3 + \frac{x}{4}\right)^5$.
12. $(4a^{\frac{1}{2}} - 3b^{\frac{1}{3}})^7$.

Find the general or $(r + 1)$ th terms of:

13. $(3 - 5x)^{-\frac{1}{2}}$. 14. $\left(ax + \frac{b}{x}\right)^n$. 15. $\left(a^{\frac{1}{p}} - b^{\frac{1}{q}}\right)^{-1}$.

16. If ${}^mC_p$ represent the p th coefficient of the expansion of $(a + x)^m$, show that ${}^mC_p + {}^mC_{p+1} = {}^{m+1}C_{p+1}$.

17. If P be the sum of the odd terms of the expansion of the binomial $(a + x)^m$, and Q the sum of the even terms, then will $P^2 - Q^2 = (a^2 - x^2)^m$.

18. In the last example, $4PQ = (a + x)^{2m} - (a - x)^{2m}$.

19. Show that $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{4^4} + \&c. = 4$.

20. If A, B are two consecutive coefficients of the expansion of $(1 + x)^n$, show that the coefficients after B are

$$B \cdot \frac{nB - A}{(n+2)A + B}, B \cdot \frac{(nB - A)(n-1 \cdot B - 2A)}{(n+2 \cdot A + B)(n+3 \cdot A + 2B)}, \&c.$$

21. Find the greatest term of the expansion of $\left(2 + \frac{1}{3}\right)^5$.

22. Find the greatest term of the expansion of $\left(1 - \frac{1}{4}\right)^{-3}$.

23. Show that

$$\sqrt[n]{a+x} = a^{\frac{1}{n}} \left\{ 1 + \frac{1}{3} \cdot \frac{x}{a+x} + \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{x^2}{(a+x)^2} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot \frac{x^3}{(a+x)^3}, \&c. \right\}$$

24. If n be a positive integer, prove that

$$1 - 2n + \frac{2n(2n-1)}{1 \cdot 2} - \dots + (-1)^{n-1} \cdot \frac{2n(2n-1) \dots (n+2)}{(n-1)!} \\ = (-1)^{n-1} \cdot \frac{2n}{2(n!)^2}$$

25. If n be an odd number, show that

$$1 + \frac{2n(2n-1)}{1 \cdot 2} + \frac{2n(n-1)(2n-2)(2n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \\ + \&c. + \frac{2n}{(n-1)! \cdot (n+1)!} = 2^{2n-2}.$$

26. Show that

$$\begin{aligned} \frac{1 \cdot 2 \cdot 3 \dots p \cdot 2^{p+q+1}}{(q+1)(q+2) \dots (p+q+1)} &= \frac{1}{p+1} + \frac{q}{(p+1)(p+2)} \\ &+ \frac{q(q-1)}{(p+1)(p+2)(p+3)} + \&c. + \frac{q(q-1) \dots 2 \cdot 1}{(p+1) \dots (p+q+1)} \\ &+ \frac{1}{q+1} + \frac{p}{(q+1)(q+2)} + \frac{p(p-1)}{(q+1)(q+2)(q+3)} \\ &+ \&c. + \frac{p(p-1) \dots 2 \cdot 1}{(q+1) \dots (p+q+1)}. \end{aligned}$$

27. If a, b, n are positive integers, and b less than $2a-1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

28. Show that the integral part of $(1 + \sqrt{3})^{2m+1}$ contains 2^{m+1} as a factor.

29. Find the coefficient of x^m in the expansion of

$$\frac{p+qx}{q+rx}$$

in a series of ascending powers of x , and show that if P and Q be the coefficients of x^m , according as it is expanded according to ascending or descending powers respectively;

$$\text{then } PQ = - \frac{(pr - q)^2}{q^2 r^2}.$$

30. Prove that the coefficient of x^r in

$$(1 + 2x + 3x^2 + \&c. \text{ ad inf.})^2$$

$$\text{is } \frac{1}{6}(r+1)(r+2)(r+3),$$

31. Show that if R is the remainder, after taking n terms of the expansion of $(1-x)^{-2}$, then

$$R = \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}.$$

32. Find the middle term of $(x + \frac{1}{x})^{2n}$, and show that the numerator contains 2^n as a factor.

33. Show that $\frac{(1-x)^n}{\lfloor n} = \frac{1}{\lfloor n} + \frac{1}{\lfloor n-1} \cdot \frac{x}{1-x}$
 $+ \frac{1}{\lfloor 2 \cdot \lfloor n-2} \cdot \frac{x^2}{(1-x)^2} + \&c.$

34. Prove that $(x + \frac{1}{x})^{2n}$
 $= (x^{2n} + \frac{1}{x^{2n}}) + 2n(x^{2n-2} + \frac{1}{x^{2n-2}})$
 $+ \frac{2n(2n-1)}{1 \cdot 2}(x^{2n-4} + \frac{1}{x^{2n-4}}) + \&c. + 2^n \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{\lfloor n}$

35. When x is small, show that $\frac{(1+x)^{\frac{1}{2}} + (1-2x)^{\frac{1}{2}}}{(1-3x)^{\frac{1}{2}}}$
 $= 2 + \frac{3}{2}x$, nearly.

36. If x be greater than a , prove that the sum of all the terms of the expansion of $(x+a)^n$ but the first two is less than $(2^n - n - 1)ax^{n-1}$.

CHAPTER XVII.

INDETERMINATE COEFFICIENTS AND PARTIAL FRACTIONS.

Indeterminate Coefficients.

148. If $a + bx + cx^2 + dx^3 + \&c. = 0$ for all values of x , then each of the quantities $a, b, c, d, \&c.$, must be zero.

Since for all values of x it is true that

$$a + bx + cx^2 + dx^3 + \&c. = 0 \dots \dots \dots (1),$$

it must be true when $x = 0$.

Putting, then, $x = 0$, we have $a = 0$.

Hence, from (1), $bx + cx^2 + dx^3 + \&c. = 0$;
 or, dividing each side by x ,

$$b + cx + dx^2 + \&c. = 0.$$

Since this is also true for all values of x , we may put $x = 0$.

We then get $b = 0$.

And so we may show that $c = 0$, $d = 0$, &c.

Cor. If for all values of x we have

$$A + Bx + Cx + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.;$$

then we must have $A = a$, $B = b$, $C = c$, &c.

For, transposing,

$$(A - a) + (B - b)x + (C - c)x^2 + (D - d)x^3 + \&c. = 0.$$

Hence, by the above proposition, since this is true for all values of x , we have

$$\begin{aligned} A - a &= 0, \text{ or } A = a; \\ B - b &= 0, \text{ or } B = b; \\ C - c &= 0, \text{ or } C = c; \&c. \end{aligned}$$

In many algebraical transformations the above proposition is especially useful.

Ex. 1. Given $x = y - 3y^3$, find y in a series of ascending powers of x .

$$\text{Here } x = y - 3y^3 \dots \dots \dots (1).$$

$$\text{Assume } y = ax + bx^2 + cx^3 + dx^4 + \&c. \dots \dots \dots (2).$$

We have not introduced a constant term into the series for y , for from (1) we see that, when $x = 0$, $y = 0$.

Squaring (2), we have

$$\begin{aligned} y^2 &= a^2x^2 + b^2x^4 + \&c. \\ &\quad + 2abx^3 + 2acx^4 + \&c. \\ &= a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \&c. \dots \dots \dots (3). \end{aligned}$$

Hence, substituting in (1) from (2) and (3),

$$\begin{aligned} x &= (ax + bx^2 + cx^3 + dx^4 + \&c.) \\ &\quad - 3 \{ a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \&c. \} \\ &= ax + (b - 3a^2)x^2 + (c - 6ab)x^3 + (a - 3b^2 - 6ac)x^4 + \&c. \end{aligned}$$

Equating the coefficients of like powers of x on each side of this equation, we have

$$a = 1.$$

$$b - 3a^2 = 0, \text{ or } b = 3 \times 1^2 = 3.$$

$$c - 6ab = 0, \text{ or } c = 6 \times 1 \times 3 = 18.$$

$$d - 3b^2 - 6ac = 0, \text{ or } d = 3 \times 3^2 + 6 \times 1 \times 18 = 135.$$

&c.

Substituting these values of a, b, c, d , &c., in (2), we get

$$y = x + 3x + 18x^2 + 135x^3 + \&c.$$

Ex. 2. Express the quotient of $\frac{1 + 5x}{1 - 2x + 7x^2}$ in ascending powers of x .

$$\text{Let } \frac{1 + 5x}{1 - 2x + 7x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c. \dots (1).$$

Clearing of fractions, we have

$$\begin{aligned} 1 + 5x &= a + bx + cx^2 + dx^3 + ex^4 + \&c. \\ &\quad - 2ax - 2bx^2 - 2cx^3 - 2dx^4 - \&c. \\ &\quad + 7ax^2 + 7bx^3 + 7cx^4 + \&c. \\ &= a + (b - 2a)x + (c - 2b + 7a)x^2 + (d - 2c + 7b)x^3 \\ &\quad + (e - 2d + 7c)x^4 + \&c. \end{aligned}$$

Hence, equating coefficients of like powers of x , we have

$$a = 1$$

$$b - 2a = 5, \text{ or } b = 5 + 2 \times 1 = 5.$$

$$c - 2b + 7a = 0, \text{ or } c = 2 \times 7 - 7 \times 1 = 7.$$

$$d - 2c + 7b = 0, \text{ or } d = 2 \times 7 - 7 \times 5 = -35.$$

$$e - 2d + 7c = 0, \text{ or } e = 2(-35) - 7 \times 7 = -119.$$

&c.

Hence, by substitution in (1), we have

$$\frac{1 + 5x}{1 - 2x + 7x^2} = 1 + 7x + 7x^2 - 35x^3 - 119x^4 - \&c.$$

Ex. 3 Find the square root of $1 - x + x^2$.

$$\text{Let } \sqrt{1 - x + x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c. (1).$$

$$\begin{aligned} \therefore 1 - x + x^2 &= a^2 + b^2x^2 + c^2x^4 + \&c. \\ &\quad + 2abx + 2acx^2 + 2adx^3 + 2aex^4 + \&c. \\ &\quad + 2bcx^3 + 2bdx^4 + \&c. \end{aligned}$$

Hence, equating coefficients of like powers of x , we have

$$a^2 = 1, \therefore a = 1.$$

$$2ab = -1, \therefore b = -\frac{1}{2a} = -\frac{1}{2}.$$

$$b^2 + 2ac = 1, \therefore c = \frac{1 - b^2}{2a} = \frac{3}{8}.$$

$$2ad + 2bc = 0, \therefore d = -\frac{bc}{a} = -\frac{3}{16}.$$

$$c^2 + 2ae + 2bd = 0, \therefore e = -\frac{c^2 + 2bd}{2a} = -\frac{21}{64}, \&c.$$

Hence, substituting in (1),

$$\sqrt{1-x+x^2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 - \frac{21}{64}x^4 + \&c.$$

Partial Fractions.

149. When the denominator of an algebraical fraction can be broken up into factors it is always possible to separate the fraction into two or more fractions having denominators of lower dimensions. The limits of this work will only allow space for one or two examples. We shall not enter upon the general theory.

Ex. 1. Resolve $\frac{3x+7}{x^2+x-30}$ into partial fractions.

Since $x^2 + x - 30 = (x-5)(x+6),$

assume $\frac{3x+7}{x^2+x-30} = \frac{A}{x-5} + \frac{B}{x+6} \dots\dots\dots(1).$

Clearing of fractions, then

$$3x+7 = A(x+6) + B(x-5) \dots\dots\dots(2).$$

We may now proceed by either of the following methods:

I. First Method.

Collecting like terms, we have

$$3x+7 = (A+B)x + (6A-5B).$$

Equating coefficients of like powers of x , then

and $\left. \begin{array}{l} A + B = 3 \\ 6A - 5B = 7 \end{array} \right\}$, from which $A = 2, B = 1$.

Hence, from (1) by substitution,

$$\frac{3x + 7}{x^2 + x - 30} = \frac{2}{x - 5} + \frac{1}{x + 6}$$

II. Second Method.

To find A , we shall in (2) make the original denominator of A , viz. $x - 5$, equal to zero. And similarly for B .

Thus we have in (2)

$$3x + 7 = A(x + 6) + B(x - 5).$$

Put $x - 5 = 0$, or $x = 5$, then

$$3 \times 5 + 7 = A(5 + 6), \text{ or } A = 2.$$

Put $x + 6 = 0$, or $x = -6$, then

$$3(-6) + 7 = B(-6 - 5), \text{ or } B = 1.$$

Hence, as before,

$$\frac{3x + 7}{x^2 + x - 10} = \frac{2}{x - 5} + \frac{1}{x + 6}.$$

It is often more convenient to follow the second method.

Ex. 2. Resolve $\frac{1}{(x - 1)^2 (x + 2) (x + 3)}$ into partial fractions.

Assume

$$\frac{1}{(x - 1)^2 (x + 2) (x + 3)} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{C}{x + 2} + \frac{D}{x + 3} \dots (1).$$

Then, clearing of fractions,

$$1 = A(x + 2)(x + 3) + B(x - 1)(x + 2)(x + 3) + C(x - 1)^2(x + 3) + D(x - 1)^2(x + 2) \dots (2).$$

Put $x - 1 = 0$, or $x = 1$, then

$$1 = A(1 + 2)(1 + 3), \text{ or } A = \frac{1}{12}.$$

Put $x + 2 = 0$, or $x = -2$, then

$$1 = C(-2 - 1)^2(-2 + 3) = 9C, \text{ or } C = \frac{1}{9}.$$

Put $x + 3 = 0$, or $x = -3$, then

$$1 = D(-3-1)^2(-3+2) = -16D, \text{ or } D = -\frac{1}{16}.$$

We have thus determined A, C, D.

We cannot thus find B, but shall proceed as follows:

In (1) substitute the values of A, C, D just found, and

transpose; then

$$\frac{B}{x-1} = \frac{1}{(x-1)^2(x+2)(x+3)} - \frac{1}{12(x-1)^2} - \frac{1}{9(x+2)} + \frac{1}{16(x+3)};$$

or, simplifying,

$$= \frac{x-7}{144(x-1)^2(x+2)(x+3)} = -\frac{7}{144(x-1)}.$$

$$\therefore B = -\frac{7}{144}.$$

We have therefore from (2), by substituting in (1) the values of A, B, C, D, obtained

$$\frac{1}{(x-1)^2(x+2)(x+3)} = \frac{1}{12(x-1)^2} - \frac{7}{144(x-1)} + \frac{1}{9(x+2)} - \frac{1}{16(x+3)}.$$

Ex. 3. Find the partial fractions of $\frac{3x+5}{x^4+x^2+1}$.

The factors of the denominator are x^2+x+1 and x^2-x+1 .

We shall assume the numerators of the required partial fractions to be of the form $Ax+B$ and $Cx+D$ respectively.

We then have

$$\frac{3x+5}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \dots\dots(1).$$

$$\text{Or } 3x+5 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1) \dots\dots(2).$$

Let $x^2+x+1=0$, or $x^2=-x-1$, then from (2)

$$\begin{aligned} 3x+5 &= (Ax+B)(-x-1-x+1) = -2Ax^2-2Bx \\ &= -2A(-x-1)-2Bx \\ &= (2A-2B)x+2A. \end{aligned}$$

Equating coefficients of like powers of x , then

$$2A = 5, \text{ or } A = \frac{5}{2};$$

and $2A - 2B = 3$, or $2B = 5 - 3 = 2$, or $B = 1$.

And similarly we find $C = -\frac{1}{2}$, $D = 4$.

Hence, from (1),

$$\frac{3x + 5}{x^4 + x^2 + 1} = \frac{5x + 2}{2(x^2 + x + 1)} - \frac{5x - 8}{2(x^2 - x + 1)}$$

Ex. XXXIV.

1. Find x in a series of ascending powers of y , when $y = 2x - 5x^2$.

2. Find by the method of indeterminate coefficients the quotient of $\frac{1 - 2x}{(1 - x)^3}$.

3. Find the cube root of $1 + 3x + 4x^2$.

4. Express by a series the difference between $(1 - 3x)^{-1}$ and $(1 + 2x)^{-1}$.

5. Find the sum of the series $1^3, 2^3, 3^3$, &c., to n terms.

6. Show that $1^3 + 2^3 + 3^3 + 4^3 + \&c. + x^3 = \left\{ \frac{n(n+1)}{2} \right\}^2$.

Resolve into partial fractions :

7. $\frac{x + 9}{(x + 3)(x + 5)}$.

8. $\frac{1}{(x^2 - 6)(x^2 + 8)}$.

9. $\frac{1}{x^3 + 6x^2 + 11x + 6}$.

10. $\frac{1}{x(1+x)^2(1+x+x^2)}$.

11. $\frac{4x^2 + x - 1}{3x^3 - 4x^2 + x}$.

12. $\frac{3}{1 + x^3}$.

13. $\frac{1}{(x^4 + x)(x^4 - 1)}$.

14. $\frac{1}{(x + a)(x + b)(x + c)}$.

15. $\frac{x}{(x + a)(x + b)(x + c)}$.

16. $\frac{x^2}{(x + a)(x + b)x + c}$.

17. $\frac{1}{x^3 + 6x^2 + 16x - 21}$.

18. $\frac{x^2 + mx + n}{(x + a)(x + b)(x + c)}$.

CHAPTER XVIII.

LOGARITHMS.

150. DEF.—The **logarithm** of a number to a given base is the index of the power to which the base must be raised to obtain the number.

Thus, we may obtain the numbers 1, 10, 100, 1,000, 10,000, &c., by raising the base 10 to the powers 0, 1, 2, 3, 4, &c., respectively; and hence, by the above definition, we have—

$$\log_{10} 1 = 0, \log_{10} 10 = 1, \log_{10} 100 = 2, \log_{10} 1,000 = 3, \&c.,$$

the suffix 10 being added to the word *log* to indicate that the base is 10. It is usual, however, in common logarithms to omit this suffix; and hence, when there is no base expressed, the student will understand 10.

Again, the numbers 1, 2, 4, 8, 16, &c., may be obtained by finding the values of $2^0, 2^1, 2^2, 2^3, 2^4$, &c., respectively, and hence we have by definition—

$$\log_2 1 = 0, \log_2 2 = 1, \log_2 4 = 2, \log_2 8 = 3, \&c.$$

$$\text{So also we find } \log_4 16 = 2, \log_5 125 = 3, \log_3 81 = 4, \&c.$$

Ex. Find $\log_4 256$, $\log_{36} 216$, and the logarithm of 9 to base $\sqrt{3}$.

$$\log_4 256 = \log_4 4^4 = 4, \text{ by definition.}$$

$$\log_{36} 216 = \log_{36} 6^3 = \log_{36} (6^2)^{\frac{3}{2}} = \log_{36} 36^{\frac{3}{2}} = \frac{3}{2}, \text{ by definition.}$$

$$\log_{\sqrt{3}} 9 = \log_{\sqrt{3}} 3^2 = \log_{\sqrt{3}} (\sqrt{3})^4 = 4, \text{ by definition.}$$

Characteristics of Ordinary Logarithms.

151. DEF.—The **characteristic** of a logarithm is the integral part of the logarithm, and the fractional part (generally expressed as a decimal) is called the **mantissa**.

1. *Numbers containing integer digits.*

Every number containing n digits in its integral part must lie between 10^{n-1} and 10^n .

Thus, 6 lies between 10^0 and 10^1 , 29 lies between 10^1 and 10^2 , 839 lies between 10^2 and 10^3 , &c.

Hence the ordinary logarithms of all numbers having n integer digits lies between $(n - 1)$ and n .

The integral portion or characteristic of the logarithm of a number having n integer digits is therefore $(n - 1)$.

Hence we have the following rule:—

RULE 1.—*The characteristic of the logarithm of a number having integer digits is ONE less than the number of integer digits.*

Thus, the characteristics of the logarithms of 32, 713·54, 8·7168, 56452, 73607·9 are respectively 1, 2, 0, 4, 4.

2. *Numbers less than unity expressed as decimals.*

All such numbers having n zeros immediately after the decimal point lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, or between 10^{-n} and $10^{-(n+1)}$.

Thus, ·3 lies between 1 and ·1, or between 1 and $\frac{1}{10}$, or 10^0 and 10^{-1} ;

·027 lies between ·1 and ·01, or between $\frac{1}{10}$ and $\frac{1}{10^2}$, or 10^{-1} and 10^{-2} ;

·000354 lies between ·001 and ·0001, or between $\frac{1}{10^3}$ and $\frac{1}{10^4}$, or 10^{-3} and 10^{-4} , and so on.

Hence, by Def., Art. 150, the logarithm of any number having n zeros immediately after the decimal point lies between $-n$ and $-(n + 1)$. Hence, the logarithm is *negative*, and the integral part of this negative quantity is n . It is however usual to write all the mantissæ of logarithms as *positive* quantities, and the *negative integral* part of the logarithm will be the next higher negative integer, viz., $-(n + 1)$.

We have therefore the following rule:—

RULE 2.—*The characteristic of the logarithm of a number less than unity, and expressed as a decimal, is the negative*

integer next greater than the number of zeros immediately after the decimal point.

Thus, the characteristics of the logarithms of .3, .0076, .02535, .7687, are respectively - 1, - 3, - 2, - 1.

152. *The logarithm of the PRODUCT of two numbers is the SUM of the logarithms of the numbers.*

Let m and n be the numbers, and let a be the base. Since m and n must be each some power of a , integral or fractional, positive or negative, assume—

$$\left. \begin{array}{l} m = a^x, \\ n = a^y. \end{array} \right\} \text{ Then, by definition of a logarithm,} \\ x = \log_a m, \text{ and } y = \log_a n.$$

Now we have $mn = a^x \cdot a^y = a^{x+y}$, and hence, by definition,

$$\log_a mn = x + y; \text{ we therefore have—}$$

$$\log_a (mn) = \log_a m + \log_a n. \quad Q.E.D.$$

COR. This proposition may be extended to any number of factors.

$$\text{Thus, } \log_a (mnpq) = \log_a m + \log_a n + \log_a p + \log_a q.$$

153. *The logarithm of the QUOTIENT of two numbers is found by SUBTRACTING the logarithm of the denominator from the logarithm of the numerator.*

Assuming, as in the last Art., we have—

$$x = \log_a m, y = \log_a n.$$

$$\text{Also, } \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}, \text{ and hence, by definition,}$$

$$\log_a \frac{m}{n} = x - y; \text{ we therefore have—}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n. \quad Q.E.D.$$

154. *The logarithm of the POWER of a number is found by MULTIPLYING the logarithm of the number by the INDEX of the power.*

Let it be required to find $\log_a N^p$.

Assume $N = a^x$, and therefore $x = \log_a N$.

We have $N^p = (a^x)^p = a^{px}$, and hence, by definition,
 $\log_a N^p = px = p \log_a N$. *Q.E.D.*

Ex. 1.—Given $\log 2 = .3010300$, and $\log 3 = .4771213$, find the logarithms of 18, 15, .125, 6.75.

$\log 18 = \log (2 \times 3^2) = \log 2 + 2 \log 3 = .3010300 + 2(.4771213) = 1.2552726$.

$\log 15 = \log (3 \times 5) = \log 3 + \log 10 - \log 2 = .4771213 + 1 - .3010300 = 1.1760913$.

$\log .125 = \log \left(\frac{1}{2^3}\right) = \log 1 - 3 \log 2 = 0 - 3 \times .3010300$
 $= -.9030900 = -1 + (1 - .9030900) = -1 + .0969100$,
 or, as usually written, $= \bar{1}.0969100$.

$\log 6.75 = \log \frac{27}{4} = \log \frac{3^3}{2^2} = 3 \log 3 - 2 \log 2 = 3(.4771213)$
 $- 2(.3010300) = .8293039$.

Ex. 2. Find the logarithm of $\frac{(2.4)^{\frac{1}{2}} \times (.375)^4}{(2.43)^8 \times (.032)^{-\frac{1}{2}}}$, having

given $\log 2$ and $\log 3$.

We have—

$$\begin{aligned} \log N &= \frac{1}{2} \log 2.4 + 4 \log .375 - 5 \log 2.43 \\ &\quad - (-\frac{1}{2}) \log .032. \\ &= \frac{1}{2} \log \frac{2^3 \times 3}{10} + 4 \log \frac{3}{2^3} - 5 \log \frac{3^5}{10^3} + \frac{1}{3} \log \frac{2^5}{10^3} \\ &= \frac{1}{2} (3 \log 2 + \log 3 - \log 10) + 4 (\log 3 - 3 \log 2) \\ &\quad - 5 (5 \log 3 - 2 \log 10) + \frac{1}{3} (5 \log 2 - 3 \log 10) \\ &= (\frac{3}{2} - 12 + \frac{4}{3}) \log 2 + (\frac{1}{2} + 4 - 25) \log 3 \\ &\quad + (-\frac{1}{2} + 10 - 1) \log 10 \\ &= -\frac{5}{6} \times .3010300 - \frac{41}{6} \times .4771213 + \frac{17}{6} \times 1 \\ &= -2.6590983 - 9.7809867 + 8.5 = -3.9400850 \\ &= \bar{4} + (4 - 3.9400850) = \bar{4}.0599150. \end{aligned}$$

The Use of Tables.

155. Tables have been formed of the logarithms of all numbers from 1 to 100,000, and we shall now show how they are practically used. We shall not enter here upon the method of forming the tables themselves.

The following is a specimen of the way in which the logarithms of numbers are usually tabulated:—

No.	0	1	2	3	4	5	6	7	8	9	D.	
7990	9025468	5522	5577	5631	5685	5740	5794	5848	5903	5957	54	
91	6011	6066	6120	6174	6229	6283	6337	6392	6446	6500		
92	6555	6609	6663	6718	6772	6826	6881	6935	6989	7044		
93	7098	7152	7207	7261	7315	7370	7424	7478	7533	7587		
94	7641	7696	7750	7804	7859	7913	7967	8022	8076	8130		
95	8185	8239	8293	8348	8402	8456	8511	8565	8619	8674		
96	8728	8782	8836	8891	8945	8999	9054	9108	9162	9217		
97	9271	9325	9380	9434	9488	9542	9597	9651	9705	9760		
98	9814	9868	9923	9977	0031	0085	0140	0194	0248	0303		
99	9030357	0411	0466	0520	0574	0628	0683	0737	0791	0846		
8000	0900	0954	1008	1063	1117	1171	1226	1280	1334	1388		
D. { 54 P. {												
		5	11	16	22	32		38	43	49		

Thus, if the number consist of four figures only, we have simply to copy out the figures in the column headed 0, prefix a decimal point, and the proper characteristic.

Ex. $\log 7991 = 3.9026011$, $\log 7.995 = .9028185$.

When we speak of a number consisting of four figures only, we include such numbers as .003654, .07682, &c., the number of zeros immediately following the decimal points not being counted.

Thus, $\log .07997 = \bar{2}.9029271$

$\log .007992 = \bar{3}.9026555$.

When the number contains five figures, as, for instance, 79936, we look along the line containing the first four figures—viz, 7993—of the number until the eye rests upon the column headed 6, the fifth figure. We then take the first three figures of the column headed 0, and affix the four figures of the column headed 6 in the horizontal line of the first four figures of the number.

Thus, $\log 79936 = 4.9027424$

$\log .079927 = \bar{2}.9026935$.

It will be seen from the portion of the logarithmic table above extracted, that when the first three figures of the logarithm—viz., 902—have been once printed, they are not repeated, but must be understood to belong to every four figures in each column, until they are superseded by higher figures, as 903. When, however, this change is intended to be made at any place not at the commencement of a horizontal row, the first of the four figures corresponding to the change is usually printed either in different type, or, as above, with a bar over it. Thus we have above $\bar{0}031$, indicating that from this point we must prefix 903 instead of 902.

$$\text{Thus, } \log 79\cdot986 = 1\cdot9030140,$$

$$\log \cdot0079987 = \bar{3}\cdot9030194.$$

156. *To find the logarithm of a number not contained in the tables.*

Ex. Find the logarithm of 799·1635.

Since* the *mantissa* of the number 79916·35 is the same as the *mantissa* of the given number, and that the first five figures are contained in the tables, we may proceed as follows—

(1.) Take out from the tables the mantissa corresponding to the number 79916. This is ·9026337.

(2.) Take out the mantissa of the *next higher* number in the tables—viz., 79917. This is ·9026392.

(3.) Find the difference between these mantissæ. This is called the tabular difference, being the difference of the mantissæ for a difference of *unity* in the numbers. We find tab. diff. = ·0000055, which we call D.

(4.) Then *assuming*, as will be proved in Art. 166, that small differences in numbers are proportional to the differences of the corresponding logarithms, we find the difference for ·35' = $35 \times \cdot0000055 = \cdot0000019$, retaining only 7 figures. This is often called *d*.

(5.) Now adding this value of *d* to the mantissa for the number 79916, we get the mantissa corresponding to the number 79916·35.

* Thus $\log 79916\cdot35 = \log (100 \times 799\cdot1635) = 2 + \log 799\cdot1635.$

(6.) Lastly, prefix to this mantissa the proper characteristic.

The whole operation may stand thus—

$$\text{M. of log } 79916 = \cdot 9026337 \dots\dots\dots (1).$$

$$\text{M. of log } 79917 = \cdot 9026392$$

$$\text{Tabular difference or D} = \cdot 0000055$$

Hence, difference for $\cdot 35$ or d

$$= \cdot 35 \times \cdot 0000055 = \cdot 0000019 \dots\dots\dots (2).$$

Hence, adding (1.) and (2.)—

$$\text{M. of log } 79916 \cdot 35 = \cdot 9026356.$$

$$\therefore \text{log } 799 \cdot 1635 = 2 \cdot 9026356.$$

or better thus, omitting the useless ciphers—

$$\text{M. of log } 79916 = \cdot 9026337$$

$$\text{M. of log } 79917 = \cdot 9026392$$

$$\therefore \text{D} = \begin{array}{r} 55 \\ 19 \end{array}$$

$$\text{Hence, } d = \cdot 35 \times 55 = 19$$

$$\therefore \text{M. of log } 79916 \cdot 35 = \cdot 9026356, \text{ as before.}$$

In the next article we shall show how the required difference may be obtained by inspection from the tables.

157. *Proportional parts.*

We saw in the example just worked that the tab. diff. (omitting the useless ciphers) is 55, and if we examine the table in Art. 155, we shall find the difference between the mantissæ of any two consecutive numbers there to be 54 or 55—generally 54. The number 54 is therefore placed in a separate column at the right of the table, and headed D.

The student will understand that the tab. diff. changes from time to time, and is not *always* 54 or 55.

Now assuming as in (4.) of the last article, we have—

$$\text{Diff. for } \cdot 1 = 54 \times \cdot 1 = 5 \qquad \text{Diff. for } \cdot 6 = 54 \times \cdot 6 = 32$$

$$\text{,, } \cdot 2 = 54 \times \cdot 2 = 11 \qquad \text{,, } \cdot 7 = 54 \times \cdot 7 = 38$$

$$\text{,, } \cdot 3 = 54 \times \cdot 3 = 16 \qquad \text{,, } \cdot 8 = 54 \times \cdot 8 = 43$$

$$\text{,, } \cdot 4 = 54 \times \cdot 4 = 22 \qquad \text{,, } \cdot 9 = 54 \times \cdot 9 = 49$$

$$\text{,, } \cdot 5 = 54 \times \cdot 5 = 27$$

We find therefore the numbers 5, 11, 16, 22, 27, 32, 38, 43, 49 placed in a horizontal row at the bottom marked P; in the columns respectively headed 1, 2, 3, 4, &c.

Hence, if we require the difference for (say) $\cdot 7$, we take out the number 38 from the horizontal row marked P, instead of being at the trouble to find it by actual computation.

The following example will illustrate how we proceed when we require the difference for a decimal containing more than one decimal figure. No explanation is needed.

Ex. Find $\log 7994\cdot 3726$ —

$$\begin{array}{rcl}
 \text{M. of } \log 79943 & = & \cdot 9027804 \\
 \text{Diff. for } 7 & = & 38 \\
 \text{,, } 2 & = & 11 \\
 \text{,, } 6 & = & 32 \\
 \hline
 \therefore \text{M. of } \log 79943726 & = & \cdot 9027843 \\
 \text{Hence, } \log 7994\cdot 3726 & = & 3\cdot 9027843.
 \end{array}$$

158. *Having given the logarithm of a number to find the number.*

After the explanations of Art 156, the method of working the following examples will be easily understood :—

Ex. 1. Find the number whose logarithm is $\bar{1}\cdot 9030173$. Taking from the tables the mantissæ next above and below, we have—

$$\begin{array}{l}
 \cdot 9030194 = \text{M. of } \log 79987 \\
 \cdot 9030140 = \text{M. of } \log 79986 \dots\dots\dots(1). \\
 \therefore 54 = \text{'D.} \\
 \text{Again, } \cdot 9030173 = \text{M. of } \log N \dots\dots\dots(2).
 \end{array}$$

Hence, subtracting (1) from (2)—

$33 = d$, the difference between the logarithms of the required number and the next lower.

Now $\frac{33}{54} = \cdot 61$, the difference between the next lower number and the required number.

$$\text{Hence } \cdot 9030173 = \text{M. of } \log 79986\cdot 61;$$

$$\therefore \bar{1}\cdot 9030173 = \log 7998661;$$

$\therefore 7998661$ is the number required.

$$\text{Ex. 2.* Find the value of } \frac{(1\cdot 023)^3 \times (\cdot 00123)^{\frac{1}{2}}}{(1\cdot 32756)^4}.$$

* The logarithms used in this example are taken from the tables.

We have—

$$\log N = 3 \log 1.023 + \frac{1}{4} \log .00123 - 4 \log 1.32756.$$

$$\text{Now, } 3 \log 1.023 = 3 \times .0098756 = .0276268$$

$$\begin{aligned} \frac{1}{4} \log .00123 &= \frac{1}{4} (\bar{3}.0899051) \\ &= \frac{1}{4} (4 + 1.0899051) = \bar{1}.2724763 \end{aligned}$$

$$\therefore \text{adding, } 3 \log 1.023 + \frac{1}{4} \log .00123 = \bar{1}.3001031$$

$$\begin{array}{rcl} \text{Again, M. of log 13275} & = & .1230345 \\ \text{and diff for } .6 & = & 196 \end{array}$$

$$\therefore \text{M. of log 13275.6} = .1230541$$

$$\therefore 4 \log 1.32756 = 4 \times .1230541 = .4922164$$

$$\text{Then, subtracting, log N} = \bar{2}.5078867$$

$$\begin{array}{rcl} \text{Hence we have, } .5078867 & = & \text{M. of log N,} \\ \text{and } .5078828 & = & \text{M. of log 32202;} \end{array}$$

$$\therefore \frac{39}{135} = d,$$

$$\text{also } \frac{135}{39} = D,$$

$$\text{and } \frac{39}{135} = .29.$$

$$\therefore .5078867 = \text{M. of log } 32202.29;$$

$$\therefore \bar{2}.5078867 = \log .03220229.$$

Hence .03220229 is the number required.

Ex. XXXV.

1. Find the logarithm to base 4 of the following numbers: 16, 64, 2, .25, .0625, 8.
2. Find the value of $\log_8 32$, $\log_{\sqrt[5]{6}} 25$, $\log_{.01} .729$.
3. Given $\log 2 = .3010300$, and $\log 3 = .4771213$, find the logarithms of 12, 36, 45, 75, .04, 3.75, .6, .074.
4. Given $\log 20763 = 4.3172901$, what is the logarithm of 2.0763, 2076.3, .020763, .0020763?
5. Write down the characteristics of the common logarithms of 29.6, .25402, .0034, 6176.003.
6. Given $\log 20.912 = 1.3203956$, what numbers correspond to the following logarithms:— $\bar{2}.3203956$, 6.3203956 , $\bar{1}.3203956$, 4.3203956 ?

7. Given $\log 20\cdot713 = 1\cdot3162430$, and $\log 20714 = 3\cdot3162640$, find $\log 2071457$.

8. Given $\log 3\cdot4937 = \cdot5432856$, and $\log 3\cdot4938 = \cdot5432980$, find the number whose logarithm is $3\cdot5432930$.

9. Given $\log 1\cdot05 = \cdot0211893$, $\log 2\cdot7 = 1\cdot4313638$, $\log 135 = 2\cdot1303338$, find the value of $\log \frac{(2\cdot7)^{\frac{1}{3}} \times \sqrt[5]{13\cdot5}}{(1\cdot05)^6}$.

10. Given $\log 18 = 1\cdot2552725$, and $\log 2\cdot4 = \cdot3802112$, find the value of $\log \cdot00135$.

11. What are the characteristics of $\log_3 1167$, and $\log_4 1965$?

12. Having $\log 2 = \cdot3010300$, and $\log 3 = \cdot4771213$, find x when $18^x = 125$.

13. Given $\log 47582 = 4\cdot6774427$, and $\log 47583 = 4\cdot6774518$, find $\log 47\cdot58275$.

14. Given $\log 5\cdot2404 = \cdot7193644$, and $\log 524\cdot05 = 2\cdot7193727$, find $\log 5240463$.

15. Given $\log \cdot56145 = \bar{1}\cdot7493111$, and $\log 56\cdot146 = 1\cdot7493188$, find $\log \sqrt[3]{\cdot05614581}$.

16. Given $\log 61683 = 4\cdot7901655$, and $\log 616\cdot84 = 2\cdot7901725$, find the number whose logarithm is $2\cdot7901693$.

17. Find the value of $(1\cdot05)^{15}$, having given $\log 1\cdot05 = \cdot0211893$, $\log 20789 = 4\cdot3178336$, and $\log 20790 = 4\cdot3178545$.

18. Find the compound interest of £120 for 10 years at 4 per cent. per annum, having given $\log 1\cdot04 = \cdot0170333$, $\log 14802 = 4\cdot1703204$, and $\log 14803 = 4\cdot1703497$.

19. A corporation borrows £8,630 at $4\frac{1}{2}$ per cent. compound interest, what annual payment will clear off the debt in 20 years?

$\log 1\cdot045 = \cdot0191163$, $\log 4\cdot1464 = 6176712$, and
 $\log 4\cdot1465 = \cdot6176817$.

CHAPTER XIX.

THE EXPONENTIAL THEOREM.—LOGARITHMIC SERIES.

Exponential Theorem.

159. To expand a^x in a series of ascending powers of x .

$$\begin{aligned} a^x &= \{1 + (a - 1)\}^x \\ &= 1 + x(a - 1) + \frac{x(x-1)}{1 \cdot 2} (a - 1)^2 \\ &\quad + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} (a - 1)^3 + \&c. \\ &= 1 + x(a - 1) + \frac{x^2 - x}{2} (a - 1)^2 \\ &\quad + \frac{x^3 - 3x^2 + 2x}{6} (a - 1)^3 + \&c. \end{aligned}$$

Collecting the terms involving the first power only of x , we have

$$a^x = 1 + \left\{ (a - 1) - \frac{1}{2} (a - 1)^2 + \frac{1}{3} (a - 1)^3 - \&c. \right\} x + \text{terms in } x^2, x^3, \&c.$$

Assume then

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \&c. \dots\dots\dots (1),$$

$$\text{Where } A = (a - 1) - \frac{1}{2} (a - 1)^2 + \frac{1}{6} (a - 1)^3 - \&c. \dots\dots\dots (2).$$

From (1), squaring and arranging, we have

$$a^{2x} = 1 + 2Ax + (A^2 + 2B)x^2 + (2C + 2AB)x^3 + \&c\dots\dots (3).$$

But from (1), putting therein $2x$ for x , we also get

$$\begin{aligned} a^{2x} &= 1 + A(2x) + B(2x)^2 + C(2x)^3 + \&c. \\ &= 1 + 2Ax + 4Bx^2 + 8Cx^3 + \&c. \dots\dots\dots (4). \end{aligned}$$

Hence the series in (3) and (4) are identical, and we may therefore equate the coefficients of like powers of x .

$$\text{Hence } 4B = A^2 + 2B, \text{ or } 2B = A^2. \text{ Therefore } B = \frac{A^2}{1 \cdot 2}.$$

$$8C = 2C + 2AB, \text{ or } 6C = 2A \cdot \frac{A^2}{1 \cdot 2} \therefore C = \frac{A^3}{1 \cdot 2 \cdot 3} \&c.$$

Hence, substituting in (1),

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1 \cdot 2} + \frac{A^3x^3}{1 \cdot 2 \cdot 3} + \&c. \dots \dots (5).$$

Since this equation is true for all values of x , we may put

$$Ax = 1, \text{ and } \therefore x = \frac{1}{A}.$$

$$\text{We then have } a^{\frac{1}{A}} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \&c.$$

Hence, by Art. 145, Ex. 5,

$$a^{\frac{1}{A}} = e, \text{ or } a = e^A \dots \dots \dots (6).$$

Taking the logarithm of each side of the equation to base e , we get

$$A = \log_e a \dots \dots \dots (7).$$

Hence, substituting in (5) for this value of A , we get

$$a^x = 1 + \frac{x \log_e a}{1} + \frac{(x \log_e a)^2}{1 \cdot 2} + \frac{(x \log_e a)^3}{1 \cdot 2 \cdot 3} + \&c.$$

The $(n + 1)$ th term of this series is $\frac{(x \log_e a)^n}{n}$, and the formula itself is known by the name of the **Exponential Theorem**.

COR. 1. Put $a = e$, then since $\log_e e = 1$, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

COR. 2. Since in (7) we have $A = \log_e a$, it follows at once from (2) that

$$\log_e a = (a - 1) - \frac{1}{2} (a - 1)^2 + \frac{1}{3} (a - 1)^3 - \&c.$$

Logarithmic Series.

$$160. \text{ To show that } \log_e (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \&c.$$

$$\text{and that } \log_e (1 - x) = -x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \&c.$$

In Art. 159, Cor. 1, we have

$$\log_e a = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$$

Put $1 + x$ for a , and therefore x for $a - 1$, then

$$\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \dots \dots \dots (1).$$

Now put $-x$, then

$$\log_e (1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \dots \dots \dots (2).$$

These series are not suitable for the calculation of logarithms. We proceed to obtain from them more convergent series.

161. To show that $\log_e (n + 1)$

$$= \log_e n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5} \cdot \frac{1}{(2n+1)^5} + \&c. \right\}.$$

$$\text{Now } \log_e \frac{1+x}{1-x} = \log_e (1+x) - \log_e (1-x)$$

$$\begin{aligned} &= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \&c. \right) \\ &\quad - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \&c. \right) \\ &= 2 \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c. \right) \dots \dots \dots (1). \end{aligned}$$

Since this is true for all values of x , we may put

$$\frac{1+x}{1-x} = \frac{m}{n}, \text{ and therefore also } x = \frac{m-n}{m+n}.$$

We then have $\log_e \frac{m}{n}$

$$= 2 \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \&c. \right\} (2).$$

Now put $m = n + 1$, and therefore

$$m - n = 1, \text{ and } m + n = 2n + 1;$$

Then we have $\log_e \frac{n+1}{n}$

$$= 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5} \cdot \frac{1}{(2n+1)^5} + \&c. \right\}.$$

Hence, transposing, $\log_e (n+1)$

$$= \log_e n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \cdot \frac{1}{(2n+1)^3} + \frac{1}{5} \cdot \frac{1}{(2n+1)^5} + \&c. \right\};$$

an important series, by which the logarithm of any number may be easily computed when that of the next lower is known.

162. To show that $\log_e (x+1)$

$$= 2 \log_e x - \log_e (x-1) - 2 \left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

We have in (2) of the last Art., putting

$m = x^2$, and $n = x^2 - 1$, and therefore

$m - n = 1$, and $m + n = 2x^2 - 1$,

$$\log_e \frac{x^2}{x^2-1} = 2 \left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

$$\text{But } \log_e \frac{x^2}{x^2-1} = \log_e \frac{x^2}{(x+1)(x-1)}$$

$$= 2 \log_e x - \log_e (x+1) - \log_e (x-1).$$

Hence $2 \log_e x - \log_e (x+1) - \log_e (x-1)$

$$= 2 \left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

Or, transposing, $\log_e (x+1)$

$$= 2 \log_e x - \log_e (x-1) - 2 \left\{ \frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

This is a very useful formula, by which the logarithm of any number is found from the logarithms of the two numbers next preceding.

163. To show that, when x is large, whatever be the base,

$$2 \log x = \log (x+1) + \log (x-1)$$

$$+ \frac{1}{2x} \left\{ 1 + \frac{1}{6x^2} + \frac{1}{90x^4} + \&c. \right\} \log \frac{x+1}{x-1}.$$

$$\begin{aligned}
 \text{Now, } \log_e \frac{x^2}{x^2-1} &= -\log_e \frac{x^2-1}{x^2} = -\log_e \left(1 - \frac{1}{x^2}\right) \\
 &= -\left(-\frac{1}{x^2} - \frac{1}{2} \cdot \frac{1}{x^4} - \frac{1}{3} \cdot \frac{1}{x^6} - \&c.\right) \\
 &= \frac{1}{x^2} + \frac{1}{2} \cdot \frac{1}{x^4} + \frac{1}{3} \cdot \frac{1}{x^6} + \dots \dots \dots (1).
 \end{aligned}$$

$$\begin{aligned}
 \text{And } \log_e \frac{x+1}{x-1} &= \log_e \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}, \text{ and therefore, by Art. 161 (1),} \\
 &= 2 \left\{ \frac{1}{x} + \frac{1}{3} \cdot \frac{1}{x^3} + \frac{1}{5} \cdot \frac{1}{x^5} + \&c. \right\} \dots (2).
 \end{aligned}$$

Dividing (1) by (2), we have, after performing the division,

$$\log_e \frac{x^2}{x^2-1} \div \log_e \frac{x+1}{x-1} = \frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{6x^3} + \frac{7}{90x^5} + \&c. \right\} \dots (3).$$

$$\text{Now, } \log_e \frac{x^2}{x^2-1} = 2 \log_e x - \log_e (x+1) - \log_e (x-1);$$

Hence, multiplying each side of (3) by $\log_e \frac{x+1}{x-1}$, and transposing, &c.

$$\begin{aligned}
 2 \log_e x &= \log_e (x+1) + \log_e (x-1) \\
 + \frac{1}{2x} \left\{ 1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c. \right\} \log_e \frac{x+1}{x-1} \dots \dots (4).
 \end{aligned}$$

Now it is shown (Art. 162) that

$$\log_e x = \log_a a \cdot \log_a x.$$

Hence we may replace the base e in (4) by the base a , if we introduce the factor $\log_a a$ into every term. Dividing each side of the newly-formed equation by this factor, we then get the formula in (4) expressed thus:

$$\begin{aligned}
 2 \log_a x &= \log_a (x+1) + \log_a (x-1) \\
 + \frac{1}{2x} \left\{ 1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c. \right\} \log_a \frac{x+1}{x-1}.
 \end{aligned}$$

Logarithms to base e are called Napierian logarithms, from their inventor.

It is evident that, by the series of the last articles, and by similar series, Napierian logarithms may be calculated. It will be necessary to use each series however for prime numbers only, since composite numbers may be broken up into their prime factors, and their logarithms easily obtained from those of their factors.

Ex. 1. Find $\log_2 2$, $\log_2 3$.

We have

$$\log_2 \frac{m}{n} = 2 \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \&c. \right\}.$$

Put $m = 2$, $n = 1$, and $\therefore \frac{m-n}{m+n} = \frac{2-1}{2+1} = \frac{1}{3}$.

We then have

$$\log_2 2 = 2 \left\{ \frac{1}{3} + \frac{1}{3} \left(\frac{1}{3} \right)^3 + \frac{1}{5} \left(\frac{1}{3} \right)^5 + \&c. \right\};$$

Or, reducing to decimals and adding
= .6931471...

Similarly, putting $m = 3$, $n = 2$, and $\therefore \frac{m-n}{m+n} = \frac{1}{5}$,

we have

$$\log_2 \frac{3}{2} = 2 \left\{ \frac{1}{5} + \frac{1}{3} \left(\frac{1}{5} \right)^3 + \frac{1}{5} \left(\frac{1}{5} \right)^5 + \&c. \right\};$$

or, since

$$\log_2 \frac{3}{2} = \log_2 3 - \log_2 2,$$

we have,

$$\log_2 3 = \log_2 2 + 2 \left\{ \frac{1}{5} + \frac{1}{3} \left(\frac{1}{5} \right)^3 + \frac{1}{5} \left(\frac{1}{5} \right)^5 + \&c. \right\};$$

or, performing the necessary reductions, &c.,
= 1.0986122...

Ex. 2. Find $\log_5 5$, $\log_5 10$.

Put $m = 5$, $n = 4$, and $\therefore \frac{m-n}{m+n} = \frac{1}{9}$, then since

$\log_2 \frac{5}{4} = \log_2 5 - 2 \log_2 2$, we have, by transposing,

$$\log_2 5 = 2 \log_2 2 + 2 \left\{ \frac{1}{9} + \frac{1}{3} \left(\frac{1}{9} \right)^2 + \frac{1}{5} \left(\frac{1}{9} \right)^3 + \&c. \right\};$$

or, reducing, &c.,

$$= 1.6094379...$$

Again,

$$\begin{aligned} \log_2 10 &= \log_2 (2 \times 5) = \log_2 2 + \log_2 5 \\ &= .6931471... + 1.6094379... \\ &= 2.3025850... \end{aligned}$$

Ex. 3. Find $\log_2 16205$.

$$\begin{aligned} \log_2 16205 &= \log_2 (2^3 \times 3^4 \times 5^3 + 5) \\ &= \log_2 \left\{ 2^3 \times 3^4 \times 5^3 \left(1 + \frac{1}{2^3 \times 3^4 \times 5} \right) \right\} \\ &= 3 \log_2 2 + 4 \log_2 3 + 2 \log_2 5 + \log_2 \left(1 + \frac{1}{2^3 \times 3^4 \times 5} \right). \end{aligned}$$

By substituting the values of $\log_2 2$, $\log_2 3$, $\log_2 5$, already found, and by developing $\log_2 \left(1 + \frac{1}{2^3 \times 3^4 \times 5} \right)$, the required logarithm is known.

Calculation of Common Logarithms.

164. To show that $\log_b N = \frac{1}{\log_a b} \cdot \log_a N$.

Let $N = a^x = b^y$ (1).

Then we have, from the definition of a logarithm,

$$x = \log_a N$$
(2).

$$y = \log_b N$$
(3).

Again, we have in (1), $a^x = b^y$, or taking logarithms to base a ,

$$x = y \log_a b;$$

or, substituting from (2) and (3),

$$\log_a N = \log_a N \cdot \log_a b$$

$$\therefore \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

COR. Put $b = 10$, and $a = e$, then

$$\log_{10} N = \frac{1}{\log_e 10} \cdot \log_e N.$$

Hence common logarithms may be found from Napierian logarithms by multiplying the latter by $\frac{1}{\log_e 10}$.

We call this quantity the *modulus* of the common system of logarithms; and if we represent the modulus by μ , we have

$$\mu = \frac{1}{\log_e 10} = \frac{1}{2.30258509...} = .43429448...$$

165. By multiplying by μ the series obtained in Arts. 160, 161, and putting $\mu \log_e = \log_{10}$, we have, omitting the suffix 10.

$$\log(1 + x) = \mu \left\{ x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \right\} \dots (1).$$

$$\log(1 - x) = \mu \left\{ -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \right\} \dots (2).$$

$$\log \frac{1+x}{1-x} = 2\mu \left\{ x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c. \right\} \dots (3).$$

$$\log \frac{m}{n} = 2\mu \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \&c. \right\} (4).$$

$$\log(n+1)$$

$$= \log n + 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3} \left(\frac{1}{2n+1} \right)^3 + \frac{1}{5} \left(\frac{1}{2n+1} \right)^5 + \&c. \right\} (5).$$

$$\log(x+1)$$

$$= 2 \log x - \log(x-1) - 2\mu \left\{ \frac{1}{2x^2-1} + \frac{1}{3} \left(\frac{1}{2x^2-1} \right)^3 + \&c. \right\} (6).$$

Theory of Proportional Parts.

166. To find the logarithm of a number containing $(n + 1)$ digits, when the table contains the logarithm of numbers containing n digits.

We have

$$\begin{aligned}\log(n + \delta) - \log n &= \log \frac{n + \delta}{n} = \log \left(1 + \frac{\delta}{n}\right) \\ &= \mu \left(\frac{\delta}{n} - \frac{1}{2} \cdot \frac{\delta^2}{n^2} + \frac{1}{3} \cdot \frac{\delta^3}{n^3} - \&c. \right).\end{aligned}$$

Suppose n to be an integer containing 5 figures, and δ a quantity less than unity.

We then have, since $\mu (= \cdot434\dots)$ is $< \frac{1}{2}$,

$$\mu \cdot \frac{\delta^2}{2n^2} < \frac{1}{4} \cdot \frac{1}{10000^2} \text{ and } \therefore < \cdot000000003;$$

$$\mu \cdot \frac{\delta^3}{3n^3} < \frac{1}{6} \cdot \frac{1}{10000^3} \text{ and } \therefore < \cdot0000000000002, \&c.$$

Hence, at least as far as the *seventh* place of decimals, the omission of all the terms of the above series after the first will not affect the result.

We then have

$$\log(n + \delta) - \log n = \mu \cdot \frac{\delta}{n} \dots\dots\dots(1).$$

And, similarly,

$$\log(n + 1) - \log n = \mu \cdot \frac{1}{n} = d \text{ suppose, } \dots\dots(2).$$

Then we have,

$$\log(n + \delta) - \log n = \delta d \dots\dots\dots(3).$$

Now d is the difference between the logarithms of two consecutive numbers;

And δ is the difference (less than unity) of two numbers, the logarithm of the greater of which is required.

Hence we have the following rule:

To find the logarithm of a number which lies between two consecutive numbers, multiply the difference of the given

number and the next lower in the tables by the difference of the logarithms of the numbers next above and below the given number, and add this result to the logarithm of the smaller given number.

This rule has been assumed in Chapter XVIII., to which the student is referred for practical applications.

REMARK. Since the mantissæ of all numbers, having the same digits, are the same, it follows that the above applies equally to numbers the integral part of which contains more than five digits.

167. To prove that

$$n^r - n(n-1)^r + \frac{n(n-1)}{|2|} (n-2)^r - \frac{n(n-1)(n-2)}{|3|} (n-3)^r + \&c. = |n, \text{ if } r = n, \text{ and } = 0, \text{ if } r < n.$$

We have

$$(e^x - 1)^n = e^{nx} - n \cdot e^{(n-1)x} + \frac{n(n-1)}{|2|} e^{(n-2)x} - \frac{n(n-1)(n-2)}{|3|} e^{(n-3)x} + \&c.$$

If we now expand each of the quantities e^{nx} , $e^{(n-1)x}$, &c., by the Exponential Theorem, and write down the coefficient of x^r in the whole expression, we have

Coefficient of x^r in $(e^x - 1)^n$

$$= \frac{n^r}{|r|} - n \cdot \frac{(n-1)^r}{|r|} + \frac{n(n-1)}{|2|} \cdot \frac{(n-2)^r}{|r|} - \frac{n(n-1)(n-2)}{|3|} \cdot \frac{(n-3)^r}{|r|} + \&c. \dots \dots \dots (1).$$

Again, $(e^x - 1)^n$

$$= (x + \frac{x^2}{|2|} + \frac{x^3}{|3|} + \&c.)^n$$

$$= x^n + \text{terms involving higher powers of } x \dots \dots \dots (2).$$

Hence the coefficient of x^n is *unity*; and since the expansion contains no terms with lower powers of x than the n th, we may say that

The coefficient of x^r in $(e^x - 1)^n = 1$, when $r = n$,
and $= 0$, when $r < n$, } ... (3).

Hence from (1) and (3), equating the coefficients of x^r ,

$$\frac{n^r}{1^r} - n \frac{(n-1)^r}{1^r} + \frac{n(n-1)}{1^2} \cdot \frac{(n-2)^r}{1^r} \\ - \frac{n(n-1)(n-2)}{1^3} \cdot \frac{(n-3)^r}{1^r} + \&c.$$

$$= 1, \text{ when } r = n, \\ \text{and } = 0, \text{ when } r < n.$$

$$\therefore n^r - n(n-1)^r + \frac{n(n-1)}{1^2} (n-2)^r \\ - \frac{n(n-1)(n-2)}{1^3} (n-3)^r + \&c. \\ = 1, \text{ if } r = n. \\ \text{and } = 0, \text{ if } r < n.$$

Ex. XXXVI.

Show that

$$1. \frac{e^{2x} - 1}{xe^x} = 2 \left\{ 1 + \frac{x^2}{1^3} + \frac{x^4}{1^5} + \&c. \right\}$$

$$2. \frac{e^{3x} + 1}{e^x} = 2 \left\{ 1 + \frac{x^2}{1^2} + \frac{x^4}{1^4} + \&c. \right\}$$

$$3. m^n = m + \frac{m(m-1)}{1 \cdot 2} (2^n - 2) \\ + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} (3^n - 3 \cdot 2^n + 3) + \&c.$$

$$4. n - n(n-1) + \frac{n(n-1)(n-2)}{2} - \&c. = 0.$$

$$5. n^{n-1} - n(n-1)^{n-1} + \frac{n(n-1)}{1 \cdot 2} (n-2)^{n-1} - \&c. = 0.$$

$$6. n^n - n(n-1)^n + \frac{n(n-1)}{1 \cdot 2} (n-2)^n - \&c. = 1^n.$$

$$7. n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1 \cdot 2} (n-2)^{n+1} - \&c. \\ = \frac{1}{2} n [n+1].$$

$$8. 1^n - n \cdot 2^n + \frac{n(n-1)}{1 \cdot 2} 3^n - \&c. + (-1)^n \cdot (n+1)^n \\ = (-1)^n [n].$$

$$9. 1^m + 2^m + 3^m + \&c. + n^m = \frac{n^{m+1}}{m+1} + \frac{n}{2} + m \cdot \frac{n^{m-1}}{1 \cdot 2} + \&c.$$

10. If p, q are each less than 1, show that $\frac{\log(1-p)}{\log(1-q)}$ is less than $\frac{p}{q-pq}$, and greater than $\frac{p-pq}{q}$.

11. If $u_r = r^p$, show that

$$u_1 - nu_2 + \frac{n(n-1)}{1 \cdot 2} u_3 + \&c. + (-1)^n u_{n+1} = (-1)^n [n].$$

12. Show that the limit of $\left(1 + \frac{1}{nx}\right)^x$, when x is infinite, is $e^{\frac{1}{n}}$.

13. Having given that $1 \cdot 2 \cdot 3 \dots n = \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}$, when n is increased without limit, show that

$1 \cdot 3 \cdot 5 \dots (2n-1) = 2^{n+\frac{1}{2}} \cdot n^n \cdot e^{-\frac{1}{2}}$, when n is increased without limit.

14. Hence show that the limit of

$$\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \cdot \frac{1}{2n-1} \text{ is } n^{-\frac{1}{2}},$$

when n is increased without limit.

15. Show that

$$\left(1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \&c.\right) \log_e(1+x) = x.$$

16. If $a = b - x$, show that $\log_e a$

$$= \log_e b - \frac{1}{2b-x} \left\{ \left(2 - \frac{1}{2}\right)x + \left(\frac{2}{2} - \frac{1}{3}\right)\frac{x^2}{b} + \left(\frac{2}{3} - \frac{1}{4}\right)\frac{x^3}{b^2} + \&c. \right\}.$$

17. If $a^2 + 1 = 0$, show that

$$(1.) e^{ax} + e^{-ax} = 2 \left\{ 1 + \frac{x^2}{2} + \frac{x^4}{4} + \&c. \right\}$$

$$(2.) e^{ax} - e^{-ax} = 2a \left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \&c. \right\}$$

18. If a, b, c are consecutive numbers, show that

$$\log b = \frac{1}{2} (\log a + \log c) \\ + \mu \left\{ \frac{1}{2ac+1} + \frac{1}{3} \cdot \frac{1}{(2ac+1)^2} + \frac{1}{5} \frac{1}{(2ac+1)^3} + \&c. \right\}$$

CHAPTER XX.

THE MULTINOMIAL THEOREM.

168. It is often required to expand the powers of algebraical expressions containing more than two terms. This may be done gradually by means of the Binomial Theorem, but the method is often cumbersome and inconvenient. By means of the Multinomial Theorem we are able to do this more concisely.

169. To find the general term of the expansion of $(a + b + c + d + \&c.)^n$.

Put $b_1 = b + c + d + \&c.$,

and $n = p + q_1$.

Then the $(q_1 + 1)$ th term of $(a + b_1)^n$ is

$$\frac{n(n-1) \dots (n-q_1+1)}{|q_1|} a^{n-q_1} b_1^{q_1} \\ = \frac{n(n-1) \dots (p+1)}{|q_1|} a^p b_1^{q_1} \dots \dots \dots (1),$$

when q_1 is a positive integer.

Put $c_1 = c + d + e + \&c.$;

and $q_1 = q + r_1$.

Then the $(r_i + 1)$ th term of $b_i^{q_i}$, or $(b + c_i)^{q_i}$ is

$$\frac{q_i (q_i - 1) \dots (q_i - r_i + 1)}{|r_i|} b_i^{q_i - r_i} c_i^{r_i}$$

$$= \frac{q_i (q_i - 1) \dots (q_i + 1)}{|r_i|} b_i^{q_i} c_i^{r_i} = \frac{|q_i|}{|q| |r_i|} b_i^{q_i} c_i^{r_i} \dots \dots (2.)$$

where q_i, q, r , are positive integers.

Hence, combining this with the expression in (1), where for $b_i^{q_i}$ we may substitute its general term, we find

The general term of the expansion of $(a + b + c_i)^n$

$$= \frac{n(n-1) \dots (p+1)}{|q_i|} a^p : \frac{|q_i|}{|q| |r_i|} b_i^{q_i} c_i^{r_i}$$

$$= \frac{n(n-1) \dots (p+1)}{|q| |r|} a^p b_i^{q_i} c_i^{r_i} \dots \dots \dots (3.)$$

Proceeding in this way we shall find

The general term of the expansion of $(a + b + c + d + \&c.)^n$

$$= \frac{n(n-1) \dots (p+1)}{|q| |r| \&c.} a^p b^q c^r \&c.$$

where $q, r, \&c.$, are positive integers.

Since $n = p + q$, $q = q + r_i$, $r_i = r + s_i$, $\&c.$, we must have

$p + q + r + \&c. = n$, and $q, r, s, \&c.$, positive integers.

COR. When n is a positive integer, we have

General term of $(a + b + c + d + \&c.)^n$

$$= \frac{n(n-1) \dots (p+1) \cdot p(p-1) \dots 1}{|p| |q| |r| \&c.} a^p b^q c^r \&c.$$

$$= \frac{|n|}{|p| |q| |r| \&c.} a^p b^q c^r \&c.$$

where $p + q + r + \&c. = n$.

170. To expand $(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$.

In the result of Art. 169, put $a = a_0$, $b = a_1x$, $c = a_2x^2$, $\&c.$, then

$$\begin{aligned}
 &\text{General term of } (a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n \\
 &= \frac{n(n-1)\dots(p+1)}{\begin{array}{c} |q \\ |r \\ |s \\ \&c. \end{array}} a_0^p (a_1x)^q (a_2x^2)^r (a_3x^3)^s \&c. \\
 &= \frac{n(n-1)\dots(p+1)}{\begin{array}{c} |q \\ |r \\ |s \\ \&c. \end{array}} a_0^p a_1^q a_2^r a_3^s \dots x^{q+2r+3s+\&c.} (A).
 \end{aligned}$$

where $p + q + r + s + \&c. = n$, and $q, r, s, \&c.$, *positive integers*.

When n is a positive integer, this result, by Art. 56, Cor., may be thus expressed :

$$\begin{aligned}
 &\text{General term of } (a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n \\
 &= \frac{|n|}{\begin{array}{c} |p \\ |q \\ |r \\ |s \\ \&c. \end{array}} a_0^p a_1^q a_2^r a_3^s \dots x^{q+2r+3s+\&c.} \dots (B).
 \end{aligned}$$

where $p + q + r + s + \&c. = n$, and $p, q, r, s, \&c.$, are positive integers.

Suppose now we are required to find the coefficient of the term involving x^m .

Taking the formula in (A) above, we have the conditions

$$\begin{aligned}
 p + q + r + s + \&c. &= n, \\
 \text{and } q + 2r + 3s + \&c. &= m.
 \end{aligned}$$

where $q, r, s, \&c.$, are *positive integers*.

Now, by assigning to $q, r, s, \&c.$, *all possible values*, subject to the condition that they are positive integers, and that

$$q + 2r + 3s + \&c. = m,$$

we may obtain sets of values of $p, q, r, \&c.$, which, when substituted in (A), will give the partial coefficients of x^m ; and the sum of these will give the whole coefficient.

Ex. 1. Find the coefficient of x^6 in the expansion of $(1 - 3x + 4x^2 - 2x^3)^5$.

As $n = 5$, a positive integer, we may use the formula (B).

We have

$$p + q + r + s = 5 \dots \dots \dots (1).$$

$$q + 2r + 3s = 6 \dots \dots \dots (2).$$

We shall find it convenient to proceed as follows :

1. Commence with the highest possible value of the letter

at the right of (2), viz., in this case, of s ; and take each lower value until we arrive at $s = 0$.

2. Then with the resulting equation—here with the equation containing $q + 2r$ —take first the highest possible value of r , and so on.

3. Find the value of p from (1) by means of the values of q, r, s thus obtained.

Thus :—

Take $s = 2$, and $\therefore q + 2r = 0$, where we may have
 $r = 0, q = 0, p = 3$.

Take $s = 1$, and $\therefore q + 2r = 3$, where we may have
 $r = 1, q = 1, p = 2$,

or, $r = 0, q = 3, p = 1$.

Take $s = 0$, and $\therefore q + 2r = 6$, where we may have
 $r = 3, q = 0, p = 2$,

$r = 2, q = 2, p = 1$,

$r = 1, q = 4, p = 0$.

These are all the possible solutions, and they are conveniently arranged thus :

p	q	r	s
3	0	0	2
2	1	1	1
1	3	0	1
2	0	3	0
1	2	2	0
0	4	1	0

Hence, from (B), since $a_0 = 1, a_1 = -3, a_2 = 4, a_3 = -2$, substituting the values of p, q, r, s , and omitting from the formula every term containing a letter whose value

may happen to be zero, and remembering that $\lfloor 1 = 1$, we have

$$\begin{aligned}
 & \text{Coefficient of } x^4 \\
 &= \frac{\lfloor 5}{\lfloor 3 \lfloor 2} (1)^3 (-2)^2 + \frac{\lfloor 5}{\lfloor 2} (1)^3 (-3) (4) (-2) \\
 & \quad + \frac{\lfloor 5}{\lfloor 3} (1) (-3)^3 (-2) + \frac{\lfloor 5}{\lfloor 2 \lfloor 3} (1)^2 (4)^2 \\
 & \quad + \frac{\lfloor 5}{\lfloor 2 \lfloor 2} (1) (-3)^2 (4)^2 + \frac{\lfloor 5}{\lfloor 4} (-3)^4 (4) \\
 &= 40 + 1440 + 1080 + 640 + 4320 + 1620 = 9140.
 \end{aligned}$$

Ex. 2. Find the coefficient of $x^{\frac{1}{3}}$ in $(1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}}$

Here $n = \frac{1}{3}$, a fraction; we shall therefore use the formula (A).

We have

$$p + q + r + s + \&c. = \frac{1}{3} \dots \dots \dots (1).$$

$$\text{and} \quad q + 2r + 3s + \&c. = 3 \dots \dots \dots (2).$$

The equations (1) and (2) need not contain any letter beyond s ; for if we take another letter, as t , we have

$$q + 2r + 3s + 4t = 3;$$

and the only value of t possible is $t = 0$, and so of any other letters. We shall therefore treat equations (1) and (2) as if they were written

$$\begin{aligned}
 p + q + r + s &= \frac{1}{3}. \\
 q + 2r + 3s &= 3.
 \end{aligned}$$

On trial we find the only solutions to be as in the annexed table:

p	q	r	s
$-\frac{2}{3}$	0	0	1
$-\frac{5}{3}$	1	1	0
$-\frac{8}{3}$	3	0	0

Hence the coefficient of x^3

$$\begin{aligned}
 &= \frac{1}{1}(10) + \frac{\frac{1}{3}(-\frac{2}{3})}{1}(3)(6) + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{\underline{1}3}(3)^3 \\
 &= \frac{10}{3} - 4 + \frac{5}{3} = 1.
 \end{aligned}$$

We may verify this result.

$$\text{For } 1 + 3x + 6x^2 + 10x^3 + \&c. = (1 - x)^{-3}$$

$$\begin{aligned}
 \therefore (1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}} &= \left\{ (1 - x)^{-3} \right\}^{\frac{1}{3}} \\
 &= (1 - x)^{-1} = 1 + x + x^2 + x^3 + \&c.
 \end{aligned}$$

Ex. XXXVII.

Find the coefficient of

1. x^3 in $(1 + x + x^2)^4$.
2. x^4 in $(1 - 2x + 3x^2 - 4x^3)^3$.
3. x^5 in $(4 - 5x + x^2)^{\frac{1}{2}}$.
4. x^3 in $(6 + 5x - 6x^2)^5$.
5. x^4 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.
6. x^3 in $\left(1 + \frac{1}{3}x + \frac{2}{27}x^2 + \frac{14}{81}x^3 + \&c.\right)^4$.
7. a^3b^2cd in $(a + b + c + d)^7$.

8. a^3b^2 in $(a + b + c + d)^5$.
9. abc^4d^2 in $(a + b + c + d + e)^8$.
10. x^5 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.
11. x^4 in $(1 + 2x + 3x^2 + 4x^3 + \&c.)^{-\frac{3}{2}}$.
12. x^5 in $(1 + 4x + 10x^2 + 20x^3 + 35x^4 + 42x^5 + \&c.)^{-4}$.
13. Expand $(1 - 3x + 4x^2)^{-\frac{1}{2}}$ to five terms.
14. Expand $(1 + 7x^2 + x^3)^{\frac{1}{2}}$ to four terms.
15. Show that the coefficient of the middle term of the expansion of $(1 + x + x^2)^n$

$$\begin{aligned}
 &= 1 + \frac{m(m-1)}{(\underline{1})^2} + \frac{m(m-1)(m-2)(m-3)}{(\underline{2})^2} \\
 &\quad + \frac{m(m-1)\dots(m-5)}{(\underline{3})^2} + \&c. \\
 &\quad + \frac{m(m-1)\dots(m-2r+1)}{(\underline{r})^2} + \&c.
 \end{aligned}$$

16. If c_r denote the coefficient of x^r in the expansion of $(1 + 2x + 2x^2 + 2x^3 + 2x^4 + \&c.)^n$, show that

$$(m+1)c_{m+1} - 2nc_m - (m-1)c_{m-1} = 0.$$

17. If $c_0, c_1, c_2, \&c.$, be the successive coefficients of the expansion of $(1 + 3x + 5x^2 + \&c. + 2p+1 \cdot x^p)^n$, show that

$$c_0 + c_1 + c_2 + \&c. + c_{np} = (p+1)^{2n}.$$

18. Show that the sum of the coefficients of the expansion of

$$\begin{aligned}
 &\{1 - 3x - 5x^2 + 7x^3 + 9x^4 \dots + (-1)^m (4m-1)x^{2m-1}\}^n \\
 &= 0, \text{ or } (-2)^n, \text{ according as } m \text{ is of the form } 2p \text{ or } 2p+1.
 \end{aligned}$$

$$\left. \begin{array}{l} \text{We have also } P = \frac{M}{R^n} \\ \text{and } \log P = \log M - n \log R \end{array} \right\} \dots\dots\dots(2).$$

$$\left. \begin{array}{l} \text{And again, } R = \sqrt[n]{\frac{M}{P}} \\ \text{and } \therefore \log R = \frac{1}{n}(\log M - \log P) \end{array} \right\} \dots\dots\dots(3).$$

$$\left. \begin{array}{l} \text{We have also, } R^n = \frac{M}{P} \\ \text{and } \therefore n = \frac{\log M - \log P}{\log R} \end{array} \right\} \dots\dots\dots(4).$$

COR. If n be fractional, and interest be allowed for the fractional part of a year at the same rate, the above formulæ require modification.

For, let $n = m + \frac{1}{p}$, where $\frac{1}{p}$ is a proper fraction, and m an integer.

We have, by (1), amount for m years $= PR^m$.

Now amount of £1 for $\frac{1}{p}$ year $= 1 + \frac{r}{p}$.

Hence, amount of PR^m for $\frac{1}{p}$ year, and

$$\begin{aligned} \therefore \text{Amount of } P \text{ for } \left(m + \frac{1}{p}\right) \text{ years} \\ = PR^m \left(1 + \frac{r}{p}\right). \end{aligned}$$

It is easy to see here that interest is really reckoned in two ways, for whole years, and for a fraction of a year. If it be agreed that interest is to be reckoned for every m th part of a year, we obtain a different formula, as shown in the next article.

173. *To find the amount of a given sum at compound interest, interest being reckoned m times a year.*

In n years there are mn distinct periods, for which com-

pound interest is to be reckoned. We therefore have to find the amount of a sum P for mn periods of time, when $\frac{r}{m}$ is the interest of £1 for *one such period*.

Hence we have simply to apply formula (1) of last article, where we must put mn for n , and $\frac{r}{m}$ for r . We then get

$$M = P \left(1 + \frac{r}{m} \right)^{mn}.$$

COR. Let m be infinite, that is, suppose interest to be considered due every instant, then

$$M = \text{limit of } P \left(1 + \frac{r}{m} \right)^{mn}, \text{ when } m = \infty : \text{ or, by Art. 42,}$$

$$M = Pe^{rn}.$$

Annuities.

174. *To find the amount of an annuity for a given time, reckoning simple interest.*

Let A be the annuity, n the number of years, r the interest of £1 for one year, M the amount.

Then, amount due at the end of

$$1 \text{ year} = A.$$

$$2 \text{ „} = A + (1+r)A.$$

$$3 \text{ „} = A + (1+r)A + (1+2r)A.$$

$$\&c. = \&c.$$

$$n \text{ „} = A + (1+r)A + (1+2r)A + \&c. + (1 + \overline{n-1} \cdot r)A.$$

$$\text{Or, } M = \{1 + (1+r) + (1+2r) + \&c. + (1 + \overline{n-1} \cdot r)\} A.$$

$$= \left\{ 2 \times 1 + \overline{n-1} \cdot r \right\} \frac{n}{2} \cdot A = nA + \frac{1}{2} n(n-1) rA.$$

175. *To find the present worth of an annuity to continue for a given time, reckoning simple interest.*

I. Let P be the present worth of an annuity A , which is to continue for n years.

Then if P be put out to simple interest for n years, its amount will be equal to the amount of the annuity for the same time.

Hence, $P(1 + nr) = nA + \frac{1}{2} n(n-1) rA$. Therefore

$$P = \frac{nA + \frac{1}{2} n(n-1) rA}{1 + nr} = \frac{nA}{2} \cdot \frac{2 + (n-1)r}{1 + nr} \dots\dots (A).$$

II. There is another way of solving this problem, which brings out a different result. It proceeds on the principle that the present worth of the annuity is the sum of the present worth of the separate annual payments.

$$\begin{aligned} \text{Thus, present worth of } A \text{ due 1 year hence} &= \frac{A}{1+r} \\ \text{'' '' 2 ''} &= \frac{A}{1+2r} \\ \text{'' '' 3 ''} &= \frac{A}{1+3r} \\ &\text{\&c.} = \text{\&c.} \\ \text{'' '' } n \text{ ''} &= \frac{A}{1+nr}. \end{aligned}$$

Now, taking P as the sum of these, we have

$$P = \left\{ \frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r} + \text{\&c.} + \frac{1}{1+nr} \right\} A \dots\dots (B).$$

If this sum be put out to simple interest for n years, we shall find that its amount is not the same as that of the annuity for the same time. The reason appears to be that in reality we are reckoning both *simple* and *compound* interest.

Thus, for any specified year, as the p th, the present value of a payment $A = \frac{A}{1+pr}$.

It is true that at *simple interest* this sum amounts to A in p years, and we may allow this sum A to remain till the n years are completed. But this is equivalent to *interest upon the interest* gained in the first p years, in addition to the interest upon the principal merely.

We shall presently find that when *compound interest* is reckoned throughout, the present worth of an annuity is the same by either method of procedure.

176. *To find the amount of an annuity, reckoning compound interest.*

We have

Amount due at the end of

$$\begin{aligned}
 1 \text{ year} &= A \\
 2 \text{ ,,} &= A + AR \\
 3 \text{ ,,} &= A + AR + AR^2 \\
 \&c. &= \&c. \\
 n \text{ ,,} &= A + AR + AR^2 + \&c. + AR^{n-1} \\
 &= A \cdot \frac{R^n - 1}{R - 1} = \frac{A}{r} (R^n - 1).
 \end{aligned}$$

$$\text{Hence } M = \frac{A}{r} (R^n - 1).$$

177. *To find the present worth of an annuity, reckoning compound interest.*

I. We shall proceed on the principle that if the present worth P be put out to compound interest for n years, it ought to amount to the same as the annuity for that time.

$$\text{Now, amount of } P \text{ in } n \text{ years} = PR^n;$$

$$\text{And, amount of annuity for } n \text{ years} = \frac{A}{r} (R^n - 1).$$

Equating these we have

$$PR^n = \frac{A}{r} (R^n - 1),$$

$$\therefore P = \frac{A}{r} (1 - R^{-n}).$$

II. We will now proceed on the principle that the present worth P is the sum of the present worths of the respective annual payments.

We have, Art. 172,

$$\text{Present worth of } A \text{ due 1 year hence} = \frac{A}{R}$$

$$\text{,, ,, 2 ,,} = \frac{A}{R^2}$$

$$\text{,, ,, 3 ,,} = \frac{A}{R^3}$$

$$\&c. = \&c.$$

$$\text{,, ,, } n \text{ ,,} = \frac{A}{R^n}$$

$$\therefore P = \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \&c. + \frac{A}{R^n} = \frac{A}{R} \cdot \frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}} = \frac{A}{r} (1 - R^{-n}).$$

We thus see that when compound interest is reckoned, we arrive at the same result, by finding the present worth on either principle.

178. *To find the present worth of an annuity to continue for ever.*

I. *Reckoning compound interest.*

We have
$$P = \frac{A}{r} (1 - R^{-n}).$$

When $n = \infty$, the limit of $R^{-n} = 0$; and then $P = \frac{A}{r}$.

Hence, the present worth of an annuity A to continue for ever $= \frac{A}{r}$.

II. *Reckoning simple interest.*

Taking the formula (A) in Art. 175, we have

$$P = \frac{nA}{2} \cdot \frac{2 + (n-1)r}{1 + nr} = \frac{nA}{2} \cdot \frac{\frac{2}{n} + (1 - \frac{1}{n})r}{\frac{1}{n} + r}.$$

Now, when $n = \infty$, the limit of

$$\frac{\frac{2}{n} + (1 - \frac{1}{n})r}{\frac{1}{n} + r} = \frac{0 + (1 - 0)r}{0 + r} = 1.$$

Hence, the limit of P , when $n = \infty$, $= \infty \cdot \frac{A}{2} = \infty$.

This result shows that, reckoning so-called simple interest, an *infinite sum* of money is required to be left, in order to insure an equal annual payment for ever.

It indicates therefore that the only correct method of computing annuities is on the compound interest principle.

We should arrive at the same result by taking formula (B) of Art. 63.

179. To find the present worth of an annuity to commence in p years, and to continue for q years.

Amount of annuity A to continue for q years

$$= \frac{A}{r} (R^q - 1) \dots\dots\dots (1).$$

Now, if P be the present worth, it will amount in $(p + q)$ years to exactly as much money as the annuity, if left to accumulate for q years.

And the amount of P in $(p + q)$ years $= P \cdot R^{p+q} \dots (2).$

Hence, equating (2) and (1), we have

$$P \cdot R^{p+q} = \frac{A}{r} (R^q - 1);$$

$$\therefore P = \frac{A}{r} \cdot \frac{R^q - 1}{R^{p+q}} = \frac{A}{r} (R^{-p} - R^{-p-q}).$$

COR. If the annuity is to continue for ever, we have $q = \infty$, and the limit of $R^{-p-q} = 0$.

Hence the present worth of a perpetual annuity which is to be entered upon in p years $= \frac{A}{r} \cdot R^{-p}.$

Ex. 1. A sum of £ a is borrowed for a period of m years, to be repaid by equal annual instalments, the first payment to be made after one year. Find the amount of the annual instalment.

Let A be the annual instalment.

Then the amount of this annual payment in m years

$$= \frac{A}{r} \{R^m - 1\}.$$

Again, if the sum a be allowed to accumulate for m years at compound interest, its amount $= a \cdot R^m$

Now, these two amounts ought to be equal.

Hence we have

$$\frac{A}{r} \{R^m - 1\} = a \cdot R^m;$$

$$\therefore A = \frac{a}{r} \cdot \frac{R^m}{R^m - 1} = \frac{a}{r} \cdot \frac{1}{1 - R^{-m}}.$$

Ex. 2. A debt a now due, is paid off in n years by a series of n payments in arithmetical progression; reckoning compound interest show that, if b be the first of the n payments, the common difference of the arithmetical progression is

$$\frac{aR^n r^n - b(R^n - 1)r}{R(R^{n-1} - 1) - (n-1)r}.$$

If we suppose the debt and the whole of the payments to be allowed to accumulate for the n years, the two amounts should be equal.

Now, amount of the debt a in n years $= aR^n \dots \dots \dots (1).$

And, amount of the n payments in the same time

$$= bR^{n-1} + (b+d)R^{n-2} + (b+2d)R^{n-3} + \&c. + (b+n-1)d.$$

where d is the common difference of the A. P.

But (Art. 130, Ex. 4) the sum of this series

$$\begin{aligned} &= \frac{bR^{n-1}}{1-R} + \frac{dR^{n-1}}{R} \cdot \frac{1 - \frac{1}{R^{n-1}}}{\left(1 - \frac{1}{R}\right)^2} - (b + n-1)dR^{n-1} \cdot \frac{\frac{1}{R^n}}{1 - \frac{1}{R}} \\ &= \frac{b}{r} \cdot R^n + \frac{d}{r^2} \cdot (R^n - R) - \frac{b + n-1 \cdot d}{r} \dots \dots \dots (2). \end{aligned}$$

(2) = (1), then

$$\frac{b}{r} \cdot R^n + \frac{d}{r^2} (R^n - R) - \frac{b + n-1 \cdot d}{r} = a \cdot R^n; \text{ or,}$$

$$\left\{ \frac{R^n - R}{r^2} - \frac{n-1}{r} \right\} d = aR^n - \frac{b}{r} (R^n - 1)$$

$$\therefore d = \frac{aR^n r^2 - b(R^n - 1)r}{R(R^{n-1} - 1) - (n-1)r}.$$

Ex. XXXVIII.

1. Find the compound interest on £530 for 12 years at $3\frac{1}{2}$ per cent.
2. In what time will a sum of money double itself at 5 per cent., compound interest?
3. Find the amount of an annuity of £50 for 10 years at $\frac{1}{4}$ per cent.
4. A corporation borrows £3,769 at $\frac{1}{4}$ per cent., to be re-

paid in 30 years by equal annual instalments. What will be the annual payment?

5. Find the equated time for £320, £390, £450, due respectively in 6, 7, 8 months, simple interest being reckoned.

6. In how many years will £1 amount to £5 at 6 per cent. per annum, compound interest?

7. A freehold bringing in £120 a year is sold for £1,920. Find the rate of interest.

8. A debt of £1,200 is to be paid out in 20 years by annual payments, increasing in A. P. If the first payment is £70, what is the last payment?

9. If interest be payable every instant, in how many years would £1 amount to £6?

10. The present value of a freehold to be entered upon in 5 years is £1,600. Find the rent, interest being reckoned at 6 per cent.

11. A property is let out on lease for a years at an annual rental of £ b , and after c years the lease is renewed on paying a fine of £ d . What is the additional rent equivalent to this fine?

12. An annuity a is found to amount in n years to the same sum as when b is put out for the same time at compound interest. Express n in terms of a , b , r , where r is the interest of one pound for one year.

CHAPTER XXII.

CONTINUED FRACTIONS.

180. Let x be any quantity, rational or irrational; and suppose that $x = a_0 + \frac{1}{x_1}$, $x_1 = a_1 + \frac{1}{x_2}$, $x_2 = a_2 + \frac{1}{x_3}$, &c.; where a_0 , a_1 , a_2 , a_3 , &c., are respectively the greatest integers in x , x_1 , x_2 , x_3 , &c.

We then have

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \&c.}}}, \text{ or, as often written,}$$

$$x = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{\&c.}$$

Such an expression is called a *continued fraction*, and it is *rational* or *irrational* according as the quotients $a_1, a_2, a_3, \&c.$, terminate or not.

The quantities $a_0, a_1, a_2, a_3, \&c.$, are called *incomplete quotients*; the first of them, a_0 , may be zero, but each of the others must at least be equal to unity.

The quantities $x_1, x_2, x_3, \&c.$, are called *complete quotients*; and when the continued fraction is made to terminate by a *complete* quotient, the resulting expression is identically equal to the given quantity x .

When, however, it is made to terminate by the successive *incomplete* quotients, we get successive approximations to the true value of x .

Thus—

(1.) We may express the true value of the given quantity x in either of the following ways:

$$a_0 + \frac{1}{x_1}, a_0 + \frac{1}{a_1 + \frac{1}{x_2}}, a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{x_3}}}, \&c.$$

(2.) We may express the corresponding approximations as follows:

$$a_0, a_0 + \frac{1}{a_1}, a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \&c.$$

COR. When the approximation to the given quantity x is expressed by terminating the continued fraction by an *incomplete* quotient, we may obtain the true value of x by substituting the corresponding *complete quotient* for the last *incomplete* quotient.

181. *Conversion of a given fraction into a continued fraction.*

Let $\frac{P}{Q}$ be the given fraction, and let the ordinary operation of finding the G. C. M. of P and Q be performed.

We will suppose the operation to stand thus :

$$\begin{array}{r} Q) P \ (a_0 \\ \underline{a_0 Q} \\ p) Q \ (a_1 \\ \underline{a_1 p} \\ q) p \ (a_2 \\ \underline{a_2 q} \\ r) \ \&c. \end{array}$$

Then we have

$$\frac{P}{Q} = a_0 + \frac{p}{Q}; \quad \frac{Q}{p} = a_1 + \frac{q}{p}; \quad \frac{p}{q} = a_2 + \frac{r}{q}; \ \&c$$

By successive substitutions we therefore get

$$\frac{P}{Q} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots \frac{1}{a_r + \dots}}}}$$

It will be presently seen (Art. 184) that the approximations obtained by terminating the continued fraction by the successive *incomplete quotients* are alternately less and greater than the given fraction; and further, that they gradually approach nearer and nearer to the true value. For this latter reason these approximations are called *converging fractions*, or *convergents*.

The convergents of $\frac{P}{Q}$ may be written as follows:

$$\frac{a_0}{1}, \quad \frac{a_0 a_1 + 1}{a_1}, \quad \frac{a_0 a_1 a_2 + a_2 + a_0}{a_1 a_2 + 1}, \ \&c.$$

If we look at the third of these, we notice :

(1.) Its *numerator* is found by multiplying the numerator

of the *second* convergent by the *third* incomplete quotient, and adding the numerator of the *first* convergent.

$$\text{Thus, } a_0 a_1 a_2 + a_2 + a_0 = (a_0 a_1 + 1) a_2 + a_0.$$

(2.) Its *denominator* is similarly formed.

$$\text{Thus, } a_1 a_2 + 1 = a_1 (a_2) + 1.$$

The existence of a general law for forming these convergents may therefore be suspected. This law is proved in the next article, and may be expressed thus:

Law of formation of the convergents.

(1.) To find the numerator of the $(n + 1)$ th convergent, multiply the numerator of the n th convergent by the next incomplete quotient, and add to the product the numerator of the $(n - 1)$ th convergent.

(2.) To find the denominator pursue a similar method.

In order, however, to apply this rule practically, it is evident that we require to have found at least *two convergents* by ordinary vulgar fractions. But if we make use of a *fictitious convergent* for the *first*, we may then apply the above rule when we know only one other.

Thus, assuming $\frac{1}{0}$ as the first convergent, we may form all

the convergents by the above law, if we only know $\frac{a_0}{1}$, the first approximation, and the successive incomplete quotients.

The accompanying table exhibits this to the eye at once.

Incomplete Quotients. }	a_0	a_1	a_2	a_3	&c.
Convergents,....	$\frac{1}{0}$	$\frac{a_0}{1}$	$\frac{a_0 a_1 + 1}{a_1}$	$\frac{a_0 a_1 a_2 + a_2 + a_0}{a_1 a_2 + 1}$	&c.
Order,.....	1st	2nd	3rd	4th	&c.

COR. If we designate by P_0, P_1, P_2, P_3 , &c., the numera-

tors, and by Q_0, Q_1, Q_2, Q_3 , &c., the denominators, of the successive convergents, we have

$$\begin{aligned} P_0 &= 1, P_1 = a_0, P_2 = a_0 a_1 + 1, P_3 = a_0 a_1 a_2 + a_2 + a_0, \text{ \&c.} \\ Q_0 &= 0, Q_1 = 1, Q_2 = a_1, Q_3 = a_1 a_2 + 1, \text{ \&c.} \end{aligned}$$

182. To show that $P_n = a_n P_{n-1} + P_{n-2}$
and that $Q_n = a_n Q_{n-1} + Q_{n-2}$

We shall assume the r th convergent to be formed according to this law, and then show that the $(r+1)$ th is formed according to the law.

By the last Art. we represent the r th convergent by $\frac{P_{r-1}}{Q_{r-1}}$.

We assume, then, that

$$\left. \begin{aligned} P_{r-1} &= a_{r-1} P_{r-2} + P_{r-3}, \text{ and } \\ Q_{r-1} &= a_{r-1} Q_{r-2} + Q_{r-3}, \end{aligned} \right\} \dots\dots\dots (1.)$$

$$\therefore \frac{P_{r-1}}{Q_{r-1}} = \frac{a_{r-1} P_{r-2} + P_{r-3}}{a_{r-1} Q_{r-2} + Q_{r-3}} \dots\dots\dots (2.)$$

Now, Art. 68, Cor., we can from this expression obtain the true value of $\frac{P}{Q}$, if we replace the incomplete quotient a_{r-1} by the corresponding complete quotient x_{r-1} .

$$\text{We then have } \frac{P}{Q} = \frac{x_{r-1} P_{r-2} + P_{r-3}}{x_{r-1} Q_{r-2} + Q_{r-3}}.$$

But, Art. 180, $x_{r-1} = a_{r-1} + \frac{1}{x_r}$; we therefore have

$$\frac{P}{Q} = \frac{(a_{r-1} + \frac{1}{x_r}) P_{r-2} + P_{r-3}}{(a_{r-1} + \frac{1}{x_r}) Q_{r-2} + Q_{r-3}} = \frac{x_r (a_{r-1} P_{r-2} + P_{r-3}) + P_{r-2}}{x_r (a_{r-1} Q_{r-2} + Q_{r-3}) + Q_{r-2}},$$

$$\text{or, by (1), } \frac{P}{Q} = \frac{x_r P_{r-1} + P_{r-2}}{x_r Q_{r-1} + Q_{r-2}}.$$

Now, replacing x_r by the corresponding incomplete

quotient a_r , we get the $(r + 1)$ th convergent. Thus we find—

$$\text{The numerator, } P_r = a_r P_{r-1} + P_{r-2};$$

$$\text{The denominator, } Q_r = a_r Q_{r-1} + Q_{r-2}$$

Hence, on the assumption that the law is true for the numerator and denominator of the r th convergent, we have shown it to be true for that of the $(r + 1)$ th convergent.

Now, we know, Art. 181, it to be true for the 3rd and 4th convergents; it is therefore true for the 5th; and hence for the 6th, &c. We conclude, then, that it is generally true.

183. To show that $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$.

We have

$$P_n = a_n P_{n-1} + P_{n-2}$$

$$Q_n = a_n Q_{n-1} + Q_{n-2}$$

We easily get

$$\frac{P_n - P_{n-2}}{Q_n - Q_{n-2}} = \frac{P_{n-1}}{Q_{n-1}}; \text{ or,}$$

$$P_n Q_{n-1} - Q_{n-1} P_{n-2} = Q_n P_{n-1} - P_{n-1} Q_{n-2}; \text{ or,}$$

$$-(P_n Q_{n-1} - Q_n P_{n-1}) = P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}$$

Or, multiplying each side by $(-1)^{n-1}$, we have

$$\begin{aligned} & (-1)^n (P_n Q_{n-1} - Q_n P_{n-1}) \\ &= (-1)^{n-1} (P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}). \end{aligned}$$

We learn from this equation that, *whatever be the value of r , the quantity $(-1)^r (P_r Q_{r-1} - Q_r P_{r-1})$ has a constant value.*

$$\begin{aligned} \text{When } r = 1, \text{ it becomes } & (-1) (P_1 Q_0 - Q_1 P_0) \\ &= - (a_0 \times 0 - 1 \times 1) = 1. \end{aligned}$$

We then have

$$\text{Constant value of } (-1)^r (P_r Q_{r-1} - Q_r P_{r-1}) = 1.$$

Put $r = n$, then we get

$$P_n Q_{n-1} - Q_n P_{n-1} = \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n. \quad Q.E.D.$$

COR. 1. Hence, dividing each side by $Q_n Q_{n-1}$, we have

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = (-1)^n \cdot \frac{1}{Q_n Q_{n-1}}.$$

COR. 2. *All converging fractions are in their lowest terms.*

For we have $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$.

Hence, if P_n and Q_n have a common measure, this measure must divide unity. And the same may be said of the common measure of P_{n-1} and Q_{n-1} , if any.

184. *Of two consecutive convergents, the one is greater and the other less than the true value of the continued fraction, and each convergent approaches more nearly to the true value than the preceding.*

Let $\frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be consecutive convergents to a continued fraction whose true value is x .

Then we have

$$x = \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}}; \text{ from which we have}$$

$$x_n = \frac{x Q_{n-1} - P_{n-1}}{P_n - x Q_n}, \text{ or } \frac{Q_n}{Q_{n-1}} x_n = \frac{x - \frac{P_{n-1}}{Q_{n-1}}}{\frac{P_n}{Q_n} - x} \dots \dots \dots (1).$$

Now Q_n, Q_{n-1}, x_n are positive quantities.

Hence, $x - \frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n} - x$ are of the same sign.

Therefore x lies between $\frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n}$, which proves the first part of the proposition.

Again (Art. 177), x_n (a complete quotient) is greater than unity.

Also, $\frac{Q_n}{Q_{n-1}}$ must be greater than unity.

Hence, from (1),
$$\frac{x - \frac{P_{n-1}}{Q_{n-1}}}{\frac{P_n}{Q_n} - x} > 1;$$

that is, the difference between x and $\frac{P_{n-1}}{Q_{n-1}}$ is greater than that between $\frac{P_n}{Q_n}$ and x .

Hence $\frac{P_n}{Q_n}$ is a nearer approximation to the true value of x than $\frac{P_{n-1}}{Q_{n-1}}$.

COR. Since the *first* convergent is $\frac{1}{0}$, and the *second* $\frac{a_0}{1}$, it follows that:

Convergents of an *odd* order form a decreasing series, and convergents of an *even* order form an increasing series, each series gradually approaching to the true value of the continued fraction.

185. To find the limits of the error in taking $\frac{P_{n-1}}{Q_{n-1}}$ as an approximation to the true value of x .

By Art. 68, we have

$$\begin{aligned} x &= \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}}; \text{ hence} \\ x - \frac{P_{n-1}}{Q_{n-1}} &= \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}} - \frac{P_{n-1}}{Q_{n-1}} \\ &= \frac{x_n (P_n Q_{n-1} - Q_n P_{n-1})}{Q_{n-1} (x_n Q_n + Q_{n-1})}, \text{ or, by Art. 70,} \\ &= (-1)^n \cdot \frac{x^n}{Q_{n-1} (x_n Q_n + Q_{n-1})} \\ &= (-1)^n \cdot \frac{1}{Q_{n-1} (Q_n + \frac{1}{x^n} Q_{n-1})}. \end{aligned}$$

Now x_n is > 1 , and therefore $\frac{1}{x_n} > 0$ and < 1 .

Hence

$$x - \frac{P_{n-1}}{Q_{n-1}} < (-1)^n \cdot \frac{1}{Q_{n-1}Q_n}, \text{ and } > \frac{1}{Q_{n-1}(Q_n + Q_{n-1})}.$$

COR. 1. Since Q_{n-1} is $< Q_n$, and $\therefore Q_{n-1}^2 < Q_{n-1}Q_n$, and $Q_{n-1}Q_n < Q_n^2$; we have

$$x - \frac{P_{n-1}}{Q_{n-1}} < \frac{(-1)^n}{Q_{n-1}^2}, \text{ and } > \frac{(-1)^n}{2Q_n^2}.$$

COR. 2. Hence if it be required to find a convergent $\frac{P_{n-1}}{Q_{n-1}}$ which differs from x by a quantity less than any given quantity $\frac{1}{a}$, we must find the successive convergents until we reach one whose denominator Q_{n-1} is $\equiv \sqrt{a}$.

186. Any convergent $\frac{P_n}{Q_n}$ approaches more nearly to the value of x than any other fraction whose denominator is $< Q_n$.

Let $\frac{p}{q}$ be any fraction which lies between $\frac{P_n}{Q_n}$ and x .

If $\frac{p}{q}$ be a convergent, then (Art. 184) q is necessarily $> Q_n$.

But if $\frac{p}{q}$ be not one of the convergents, then, since x lies between $\frac{P_n}{Q_n}$ and $\frac{P_{n-1}}{Q_{n-1}}$, and $\frac{p}{q}$ lies between $\frac{P_n}{Q_n}$ and x , it follows that

$$\frac{p}{q} \text{ lies between } \frac{P_n}{Q_n} \text{ and } \frac{P_{n-1}}{Q_{n-1}}.$$

$$\text{Hence } \frac{p}{q} \approx \frac{P_{n-1}}{Q_{n-1}} < \frac{P_n}{Q_n} \approx \frac{P_{n-1}}{Q_{n-1}} < \frac{P_n Q_{n-1} \approx Q_n P_{n-1}}{Q_n Q_{n-1}}.$$

But (Art. 183) neglecting the sign, $P_n Q_{n-1} \approx Q_n P_{n-1} = 1$;

$$\therefore \frac{p}{q} \approx \frac{P_{n-1}}{Q_{n-1}} < \frac{1}{Q_n Q_{n-1}}, \text{ or}$$

$$pQ_{n-1} \approx qP_{n-1} < \frac{q}{Q_n} \dots\dots\dots (1).$$

Now $pQ_{n-1} \approx qP_{n-1}$ cannot be equal to zero, for then $\frac{p}{q} = \frac{P_{n-1}}{Q_{n-1}}$, which is contrary to our hypothesis. And hence, since p, q, P_{n-1}, Q_{n-1} are all integers,

$$pQ_{n-1} \approx qP_{n-1} \text{ cannot be less than unity.}$$

Therefore the condition in (1) is impossible, if $q < Q_n$.

Hence no fraction $\frac{p}{q}$ can approach more nearly to the value of x , than the convergent $\frac{P_n}{Q_n}$, if q is $< Q_n$.

187. To solve the indeterminate equation

$$ax + by = c$$

by means of a continued fraction.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding the last (viz., $\frac{a}{b}$). Then we have (Art. 183),

$$aq - bp = \pm 1, \text{ or } a(\pm cq) - b(\pm cp) = c.$$

Or, adding and subtracting abt , we have

$$a(bt \pm cq) + b(\pm cp - at) = c.$$

Comparing this with the original equation, we see that the given equation is solved by putting

$$\begin{aligned} x &= bt \pm cq \\ y &= \mp cp - at. \end{aligned}$$

Ex. Reduce $\frac{314159}{100000}$ to a continued fraction and find the convergents.

100000	314159 300000	3
14159	100000 99113	7
887	14159 887	15
	5289 4435	
854	887 854	1
33	854 66	25
	194 165	
29	33 29	1
4	29 28	7
1	4 4	4

The continued fraction is

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{25 + \frac{1}{1 + \frac{1}{7 + \frac{1}{4}}}}}}}$$

And the convergents

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}, \frac{76149}{24239}, \frac{314159}{100000}$$

Ex. XXXIX.

Reduce to continued fractions :

1. $\frac{1}{1\frac{1}{1\frac{1}{1}}}$. 2. $\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1}}}}$ 3. $\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1}}}}}$. 4. $\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1\frac{1}{1}}}}}}$.

5. Why is $1\frac{1}{1\frac{1}{1}}$ a nearer approximation to π than $\frac{22}{7}$?

6. Show that $\frac{P_n}{Q_n} = \frac{P_{n+1} - P_{n-1}}{Q_{n+1} - Q_{n-1}}$.

7. If a_{n-1} be the n th incomplete quotient, prove that

$$a_n = \sqrt{\frac{(P_n - P_{n-2})(Q_n - Q_{n-2})}{P_{n-1} \cdot Q_{n-1}}}.$$

8. If $\frac{P_r}{Q_r}$ represents the $(r+1)$ th convergent, show that

$$(P_n Q_{n-1} - Q_n P_{n-1})(P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}) \dots \text{to } n \text{ factors} \\ = (-1)^{\frac{1}{2}n(n+1)}.$$

9. If $\frac{P_r}{Q_r}$ represents the $(r+1)$ th convergent to the fraction $\frac{P}{Q}$, show that the $(n+1)$ th complete quotient

$$= \frac{PQ_{n-1} - QP_{n-1}}{QP_n - PQ_n}.$$

10. Prove that $\frac{P_{2n}}{Q_{2n}} - \frac{P_n}{Q_n} =$

$$\frac{1}{Q_{2n}Q_{2n-1}} - \frac{1}{Q_{2n-1}Q_{2n-2}} + \frac{1}{Q_{2n-2}Q_{2n-3}} - \dots + \\ (-1)^{n-1} \cdot \frac{1}{Q_{n+1}Q_n}.$$

11. If $\frac{p}{q}$ be a convergent to a given quantity x , show that no fraction having a less denominator than q can be a nearer approximation to x than $\frac{p}{q}$.

12. Solve in positive integers the indeterminate equation $3x - 5y = 11$.

Reduction of Quadratic Irrational Quantities to Continued Fractions.

188. Every quadratic equation with rational coefficients may be reduced to the form

$$ax^2 + 2bx + c = 0,$$

where a, b, c are integers, either positive or negative.

Solving this equation, we get

$$x = \frac{\pm \sqrt{b^2 - ac} - b}{a}, \text{ which may be also}$$

written
$$x = \frac{\sqrt{b^2 - ac} \mp b}{\pm a}.$$

we put $N = b^2 - ac$, we have

$$\frac{N - b^2}{a} = -c, \text{ and } x = \frac{\sqrt{N} \mp b}{\mp a}.$$

Hence, since for the purposes of continued fractions we need consider only the *magnitude* of these roots, and since, too, we may so choose the sign of a that x shall be always positive, we arrive at the following result:

The roots (neglecting the sign) of a quadratic equation having rational coefficients may, when irrational, be always referred to the form

$$x = \frac{\sqrt{N} + p}{q},$$

where

(1.) p and q are integers either positive or negative, the sign of q being so taken that x shall be positive;

(2.) $\frac{N - p^2}{q}$ is an integer.

Continued fractions developed from such irrational quantities are called irrational continued fractions of the second degree.

189. To express $\frac{\sqrt{N} + p_0}{q_0}$ as a continued fraction, p_0 and q_0 being integers, and the sign of q_0 being so taken that the expression shall be positive, and where also $\frac{N - p_0^2}{q_0}$ is an integer.

Let a_0 be the greatest integer in $\frac{\sqrt{N} + p_0}{q_0} = x$, suppose.

$$\begin{aligned} \text{Then } x &= a_0 + \left(\frac{\sqrt{N} + p_0}{q_0} - a_0 \right) = a_0 + \frac{\sqrt{N} - (a_0 q_0 - p_0)}{q_0} \\ &= a_0 + \frac{N - (a_0 q_0 - p_0)^2}{q_0 \{ \sqrt{N} + (a_0 q_0 - p_0) \}} \\ &= a_0 + \frac{N - (a_0 q_0 - p_0)^2}{q_0 \sqrt{N} + (a_0 q_0 - p_0)} \\ &= a_0 + \frac{q_1}{\sqrt{N} + p_1} \dots \dots \dots (1), \end{aligned}$$

$$\text{where } q_1 = \frac{N - (a_0 q_0 - p_0)^2}{q_0},$$

$$\text{and } p_1 = a_0 q_0 - p_0.$$

$$\text{We then have } x = a_0 + \frac{1}{\frac{\sqrt{N} + p_1}{q_1}} = a_0 + \frac{1}{x_1} \dots \dots \dots (2),$$

$$\text{where } x_1 = \frac{\sqrt{N} + p_1}{q_1}.$$

We may notice that x_1 has the *same form* as the given quantity x ; and we may therefore in the same manner obtain as complete quotients x_2, x_3 , &c., expressed in the *same form* as x .

$$\begin{aligned} \text{We may then write } x &= \frac{\sqrt{N} + p_0}{q_0} = a_0 + \frac{1}{x_1}, \\ x_1 &= \frac{\sqrt{N} + p_1}{q_1} = a_1 + \frac{1}{x_2}, \end{aligned}$$

$$x_2 = \frac{\sqrt{N + p_2}}{q_2} = a_2 + \frac{1}{x_3},$$

$$\&c. = \&c.$$

$$x_n = \frac{\sqrt{N + p_n}}{q_n} = a_n + \frac{1}{x_{n+1}},$$

$$\&c. = \&c.$$

where $a_0, a_1, a_2, \&c.$, are all positive integers; and where $x_1, x_2, x_3, \&c.$, are also positive quantities greater than unity.

We hence have

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots \frac{1}{a_n + \dots}}}}$$

the required continued fraction.

Ex. 1. Reduce $\frac{\sqrt{17 + 3}}{4}$ to a continued fraction.

The expression is positive, p_0 and q_0 are integers, and $\frac{N - p_0}{q_0} = \frac{17 - 9}{4} = 4$.

It therefore answers the required conditions.

We have, since *unity* is the greatest integer in the given surd,

$$\begin{aligned} \frac{\sqrt{17 + 3}}{4} &= 1 + \left(\frac{\sqrt{17 + 3}}{4} - 1 \right) = 1 + \frac{\sqrt{17 - 1}}{4} \\ &= 1 + \frac{17 - 1}{4(\sqrt{17 + 1})} = 1 + \frac{4}{\sqrt{17 + 1}} \\ &= 1 + \frac{1}{\frac{\sqrt{17 + 1}}{4}} \dots \dots \dots (1). \end{aligned}$$

$$\begin{aligned} \frac{\sqrt{17 + 1}}{4} &= 1 + \left(\frac{\sqrt{17 + 1}}{4} - 1 \right) = 1 + \frac{\sqrt{17 - 3}}{4} \\ &= 1 + \frac{17 - 9}{4(\sqrt{17 + 3})} = 1 + \frac{2}{\sqrt{17 + 3}} \\ &= 1 + \frac{1}{\frac{\sqrt{17 + 3}}{2}} \dots \dots \dots (2). \end{aligned}$$

$$\frac{\sqrt{17+3}}{2} = 3 + \frac{\sqrt{17-3}}{2} = 3 + \frac{1}{\frac{\sqrt{17+3}}{4}} \dots\dots (3).$$

We have thus arrived at a quantity $\frac{\sqrt{17+3}}{4}$ exactly the same as that with which we started. The quotients will therefore recur.

We then have, by successive substitution,

$$\frac{\sqrt{17+3}}{4} = 1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{3 + \dots}}}}$$

Ex. 2. Reduce $\sqrt{26}$ to a continued fraction.

Here $p_0 = 0$, $q_0 = 1$, and the given expression answers the required conditions.

$$\sqrt{26} = 5 + (\sqrt{26} - 5) = 5 + \frac{26 - 25}{\sqrt{26} + 5} = 5 + \frac{1}{\sqrt{26} + 5}.$$

$$\text{Hence } \sqrt{26} + 5 = 10 + \frac{1}{\sqrt{26} + 5}.$$

And so, by successive substitutions, we get

$$\sqrt{26} = 5 + \frac{1}{10 + \frac{1}{10 + \frac{1}{10 + \dots}}}$$

190. To show that (i.) $p_n = a_{n-1}q_{n-1} - p_{n-1}$
(ii.) $q_nq_{n-1} = N - p_n^2$.

We have, in the last Art.

$$x_n = \frac{\sqrt{N} + p_n}{q_n} = a_n + \frac{1}{x_{n+1}} \dots\dots\dots (1).$$

Putting $n - 1$ for n , then

$$\frac{\sqrt{N} + p_{n-1}}{q_{n-1}} = a_{n-1} + \frac{1}{x_n}; \text{ from which}$$

$$x_n = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})} \dots\dots\dots (2).$$

(1) = (2), then

$$\frac{\sqrt{N} + p_n}{q_n} = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})}; \text{ or, reducing,}$$

$$N - p_n(a_{n-1}q_{n-1} - p_{n-1}) - (a_{n-1}q_{n-1} - p_{n-1} - p_n) \sqrt{N} \\ = q_n q_{n-1} - r$$

Equating the rational and irrational terms, then

$$(a_{n-1}q_{n-1} - p_{n-1} - p_n) \sqrt{N} = 0, \text{ and } \therefore p_n = a_{n-1}q_{n-1} - p_{n-1}; \\ \text{and } N - p_n(a_{n-1}q_{n-1} - p_{n-1}) = q_n q_{n-1}, \text{ or} \\ q_n q_{n-1} = N - p_n \cdot p_n = N - p_n^2.$$

The expressions (i.) and (ii.) just proved contain the law of formation of the complete quotients, and hold for all positive integral values of n .

191. *The quantities p_1, p_2, p_3 , &c., and q_1, q_2, q_3 , &c., are all integers, and moreover, after a certain value of n , p_n and q_n are positive integers.*

$$\text{We have (Art. 182)} \quad x_n = \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}}.$$

Hence also, replacing the incomplete quotient a_n by the complete quotient x_n , we get

$$x = \frac{x_n p_n + p_{n-1}}{x_n q_n + q_{n-1}} \dots \dots \dots (1).$$

$$\text{But } x = \frac{\sqrt{N} + p_0}{q_0}, \quad x_n = \frac{\sqrt{N} + p_n}{q_n}; \text{ then we have from (1)}$$

$$\frac{\sqrt{N} + p_0}{q_0} = \frac{\frac{\sqrt{N} + p_n}{q_n} \cdot p_n + p_{n-1}}{\frac{\sqrt{N} + p_n}{q_n} \cdot q_n + q_{n-1}} = \frac{p_n \cdot \sqrt{N} + p_n p_n + p_{n-1} q_n}{q_n \sqrt{N} + q_n p_n + q_{n-1} q_n}.$$

Reducing, then

$$Q_n \cdot N + p_0(Q_n p_n + Q_{n-1} q_n) + (Q_n p_0 + Q_n p_n + Q_{n-1} q_n) \sqrt{N} \\ = P_n q_0 \cdot \sqrt{N} + q_0 (P_n p_n + P_{n-1} q_n).$$

Equating the rational and irrational parts, we have

$$Q_n p_0 + Q_n p_n + Q_{n-1} q_n = P_n q_0 \dots \dots \dots (2).$$

$$Q_n \cdot N + p_0(Q_n p_n + Q_{n-1} q_n) = q_0 (P_n p_n + P_{n-1} q_n) \dots (3).$$

From (2) we have

$$Q_n p_n + Q_{n-1} q_n = P_n q_0 - Q_n p_0; \text{ and from (3)} \\ (P_n q_0 - Q_n p_0) p_n + (P_{n-1} q_0 - Q_{n-1} p_0) q_n = Q_n N.$$

Solving for q_n we get

$$(P_n Q_{n-1} - Q_n P_{n-1}) q_0 \cdot q_n = (P_n q_0 - Q_n p_0)^2 - Q_n^2 \cdot N.$$

Hence, since (Art. 183) $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$, we have

$$q_n = \frac{1}{(-1)^n q_0} \{ (P_n q_0 - Q_n p_0)^2 - Q_n^2 N \};$$

$$\text{but } \frac{1}{(-1)^n} = \frac{(-1)^n}{(-1)^n} = (-1)^n;$$

$$\therefore q_n = \frac{(-1)^n}{q_0} \{ (P_n q_0 - Q_n p_0)^2 - Q_n^2 N \} \dots\dots\dots (4).$$

And so p_n

$$= \frac{(-1)^n}{q_0} \{ Q_{n-1} Q_n N - (P_{n-1} q_0 - Q_{n-1} p_0) (P_n q_0 - Q_n p_0) \} \dots (5).$$

Now the expressions in (4) and (5) may be easily expressed thus :

$$q_n = (-1)^n \left\{ P_n^2 q_0 - 2 P_n Q_n p_0 - Q_n^2 \cdot \frac{N - p_0^2}{q_0} \right\}, \text{ and}$$

$$p_n = (-1)^n \left\{ Q_{n-1} Q_n \cdot \frac{N - p_0^2}{q_0} - P_{n-1} P_n q_0 - P_n Q_{n-1} p_0 - Q_n P_{n-1} p_0 \right\}.$$

Now $\frac{N - p_0^2}{q_0}$ is by hypothesis integral; and hence, since all the other quantities in these expressions are integers, it follows that p_n and q_n are integers. Moreover, after n has reached a certain limit, they are positive integers.

For from (2)

$$Q_{n-1} q_n = P_n q_0 - Q_n p_0 - Q_n p_n;$$

$$\therefore \frac{Q_{n-1}}{Q_n} \cdot q_n = \left(\frac{P_n}{Q_n} \cdot q_0 - p_0 \right) - p_n;$$

$$\text{but } x_n = \frac{\sqrt{N + p_n}}{q_n}, \text{ or } q_n = \frac{\sqrt{N + p_n}}{x_n}.$$

Therefore, substituting,

$$\frac{Q_{n-1}}{Q_n} \cdot \frac{\sqrt{N + p_n}}{x_n} = \left(\frac{P_n}{Q_n} q_0 - p_0 \right) - p_n.$$

$$\therefore \frac{Q_{n-1}}{Q_n} = x_n \cdot \frac{\left(\frac{P_n}{Q_n} \cdot q_0 - p_0\right) - p_n}{\sqrt{N} + p_n}.$$

Now $\frac{P_n}{Q_n}$ is the $(n+1)$ th convergent to x , or $\frac{\sqrt{N} + p_0}{q_0}$; and it differs the less from x , the greater the value of n (Art. 70).

Hence by taking n large enough $\frac{P_n}{Q_n} \cdot q_0 - p_0$ may be made to differ from $\sqrt{N}$ by as small a quantity as we please.

Hence, when n has attained a certain limit, $\frac{Q_{n-1}}{Q_n}$ differs from $x_n \cdot \frac{\sqrt{N} - p_n}{\sqrt{N} + p_n}$ by as small a quantity as we please.

Now if p_n is negative, $x_n \cdot \frac{\sqrt{N} - p_n}{\sqrt{N} + p_n}$ is greater than unity,

$$\text{and therefore } \frac{Q_{n-1}}{Q_n} > 1, \text{ or } Q_{n-1} > Q_n$$

But this is impossible, and therefore p_n cannot be negative.

Hence p_n is a *positive* integer, after n has reached a certain limit.

And so we may show that q_n is a positive integer, after n has reached a certain limit.

192. *The continued fraction is periodic after a certain number of quotients.*

For (Art. 190) we have $p_n + p_{n-1} = a_{n-1}q_{n-1} \dots \dots \dots (1).$

$$\text{and } q_n q_{n-1} = N - p_n^2 \dots \dots \dots (2).$$

Now, by the last Art., after a certain limit, p_n and q_n are positive integers.

We shall consider only the portion of the continued fraction beyond this limit.

We have then q_n, q_{n-1} , each positive integers.

Therefore from (2) $p_n < \sqrt{N}$.

and therefore, also, $p_{n-1} < \sqrt{N}$.

Hence from (1) we have $a_{n-1}q_{n-1} < 2\sqrt{N}$.

Now a_{n-1} is a positive integer, and q_{n-1} is such.

It must therefore follow that $a_{n-1} < 2\sqrt{N}$,
and $q_{n-1} < 2\sqrt{N}$.

Or, putting n for $n-1$, we find that for all values of a_n, p_n, q_n beyond a certain limit, we must have

$$p_n < \sqrt{N}, a_n < 2\sqrt{N}, q_n < 2\sqrt{N}.$$

Hence the values of a_n, p_n, q_n are limited, and the complete quotient $x_n = \frac{\sqrt{N} + p_n}{q_n}$ must have a *finite number of different values*, as must also the incomplete quotient a_n .

Therefore commencing with the first complete quotient in which p_n and q_n are positive, the values of a_n, p_n, q_n will recur, and the number of quotients in the period cannot exceed $2\sqrt{N} \times \sqrt{N}$, or $2N$.

COR. Since from (2) we have $q_n q_{n-1} < N$,
we have also $q_{n-1} q_{n-2} < N$.

Hence, if $q_{n-1} > \sqrt{N}$, we must have $q_n < \sqrt{N}$, and $q_{n-2} < \sqrt{N}$.

Hence, if any denominator of a complete quotient is $> \sqrt{N}$, the denominators immediately preceding and following are each $< \sqrt{N}$.

Ex. 1. If $N = a^2 + 1$, find $\sqrt{N}$ in the form of a continued fraction.

$$\bar{N} = a + (\sqrt{a^2 + 1} - a) = a + \frac{1}{\sqrt{a^2 + 1} + a} = a + \frac{1}{\sqrt{N} + a}.$$

Hence, by successive substitution, we get

$$\sqrt{N} = a + \frac{1}{2a + \frac{1}{2a + \dots}}.$$

Ex. 2. Develop $\frac{\sqrt{27} + 3}{2}$ as a continued fraction.

$$\frac{\sqrt{27}+3}{2} = 4 + \frac{\sqrt{27}-5}{2} = 4 + \frac{27-25}{2(\sqrt{27}+5)} = 4 + \frac{1}{\sqrt{27}+5} \dots (1)$$

$$\frac{\sqrt{27}+5}{1} = 10 + \frac{\sqrt{27}-5}{1} = 10 + \frac{27-25}{\sqrt{27}+5} = 10 + \frac{2}{\sqrt{27}+5} \dots (2)$$

$$\frac{\sqrt{27}+5}{2} = 5 + \frac{\sqrt{27}-5}{2} = 5 + \frac{27-25}{2(\sqrt{27}+5)} = 5 + \frac{1}{\sqrt{27}+5} \dots (3)$$

Hence, by successive substitutions from (1), (2), (3), we have $\frac{\sqrt{27}+3}{2} = 4 + \frac{1}{10 + \frac{1}{5 + \frac{1}{10 + \frac{1}{5 + \dots}}}}$.

193. Every periodic continued fraction is the development of one of the roots of a quadratic equation.

Suppose the period to commence after the quotient x_{m-1} , and to consist of x' terms.

Then we must have $x_m = x_{m+x'} \dots \dots \dots (1)$.

Also we have $x = \frac{x_m P_m + P_{m-1}}{x_m Q_m + Q_{m-1}} \dots \dots \dots (2)$.

And $x = \frac{x_{m+x'} P_{m+x'} + P_{m+x'+1}}{x_{m+x'} Q_{m+x'} + Q_{m+x'+1}}$.

which, by (1), $= \frac{x_m P_{m+x'} + P_{m+x'-1}}{x_m Q_{m+x'} + Q_{m+x'-1}} \dots \dots \dots (3)$.

From (2), solving for x_m , we get

$$x_m = \frac{xQ_m - P_m}{P_{m-1} - xQ_{m-1}}.$$

And from (3), solving for x_m , we get

$$x_m = \frac{xQ_{m+x'} - P_{m+x'}}{P_{m+x'-1} - xQ_{m+x'-1}}.$$

Equating these, we have

$$\frac{xQ_m - P_m}{P_{m-1} - xQ_{m-1}} = \frac{xQ_{m+x'} - P_{m+x'}}{P_{m+x'-1} - xQ_{m+x'-1}}$$

from which we get x as one of the roots of a quadratic equation.

Ex. If $x = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \dots}}}}$, express x as the root of a quadratic equation with integral coefficients.

Put $y = \frac{1}{3 + \frac{1}{4 + \dots}}$, then we have

$$x = 1 + \frac{1}{2 + y} \dots \dots \dots (1).$$

$$\text{and } x = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + y}}} \dots \dots \dots (2).$$

$$\text{From (1) } y = \frac{3 - 2x}{x - 1}; \text{ and from (2) } y = \frac{43 - 30x}{7x - 10}.$$

$$\text{Hence we have } \frac{3 - 2x}{x - 1} = \frac{43 - 30x}{7x - 10}; \text{ or, simplifying,}$$

$$16x^2 - 32x + 13 = 0, \text{ from which } x = \frac{4 + \sqrt{3}}{4}.$$

Ex. XL

Develop as continued fractions:

1. $\sqrt{10}.$ 2. $\sqrt{17}.$ 3. $\sqrt{19}.$ 4. $\sqrt{29}.$ 5. $\frac{\sqrt{14} + 3}{5}.$
6. $\frac{\sqrt{22} + 4}{3}.$ 7. $\frac{\sqrt{28} + 5}{3}.$ 8. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$

Express as continued fractions the roots of the six following equations:

9. $3x^2 - 8x + 2 = 0.$ 10. $5x^2 - 6x - 7 = 0.$
11. $2x^2 - 5x + 1 = 0.$ 12. $\frac{3}{x} = \frac{4}{x + 2} + 1.$
13. $x + \sqrt{x + 2} = 3.$ 14. $x^2 - 4ax + 3a^2 = 1.$
15. Find the value of $5 + \frac{1}{10 + \frac{1}{10 + \dots}}$.

16. Show that the product of the infinite continued fractions

$$\frac{1}{2 +} \frac{1}{3 +} \frac{1}{4 +} \frac{1}{2 + \dots}, \text{ and } 4 + \frac{1}{3 +} \frac{1}{2 +} \frac{1}{4 + \dots},$$

is $1\frac{6}{7}$.

17. Show that

$$\left(\frac{1}{m +} \frac{1}{n +} \frac{1}{p +} \frac{1}{q +} \frac{1}{m + \dots} \right) \left(q + \frac{1}{p +} \frac{1}{n +} \frac{1}{m +} \frac{1}{q + \dots} \right) \\ = \frac{n + q + npq}{m + p + mnp}.$$

18. Show that

$$\left(a + \frac{1}{2a +} \frac{1}{2a + \dots} \right) \left(a - \frac{1}{2a -} \frac{1}{2a - \dots} \right) = \sqrt{a^2 - 1}.$$

19. Find the value of $2 + \frac{1}{1 +} \frac{1}{2 +} \frac{1}{3 +} \frac{1}{2 +} \frac{1}{3 + \dots}$.

20. Find the first four convergents to the greater roots of the equations in examples 9, 10, 11.

21. Show that the roots of a quadratic which can be expressed as a recurring continued fraction are of contrary or the same sign, according as the recurring period commences with the first quotient or not.

22. When the two roots of a quadratic are developed as a continued fraction, the quotients of the recurring period occur in reverse order.

23. If $\frac{P_{n-2}}{Q_{n-2}}, \frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be successive convergents to the continued fraction

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 + \dots},$$

show that $\frac{P_n}{Q_n} = \frac{a_n P_{n-2} + b_n P_{n-1}}{a_n Q_{n-2} + b_n Q_{n-1}}$, and find whether the successive convergents are necessarily in their lowest terms.

24. If $\frac{1}{a+} \frac{1}{b+} \frac{1}{a+} \frac{1}{b+} \dots$ be a periodic continued fraction, show that

$$\frac{P_{n+2} + P_{n-2}}{P_n} = \frac{Q_{n+2} + Q_{n-2}}{Q_n} = ab + 2.$$

CHAPTER XXIII.

SERIES.

194. A series is said to be *convergent* or *divergent* according as the sum of n terms has or has not a *finite limit*, when n is increased indefinitely.

Thus, the series $1 + \frac{1}{3} + \frac{1}{9} + \&c.$ is *convergent*.

$$\text{For } S_n = \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} = \frac{3}{2} \left(1 - \frac{1}{3^n} \right).$$

$$\text{And hence, when } n = \infty, \Sigma = \frac{3}{2} (1 - 0) = \frac{3}{2}.$$

Again, the series $1 + 3 + 9 + \&c.$ is *divergent*.

$$\text{For } S_n = \frac{3^n - 1}{3 - 1} = \frac{1}{2} (3^n - 1).$$

$$\text{And, when } n = \infty, \text{ we have } \frac{1}{2} (3^n - 1) = \infty.$$

195. If all the terms of an infinite series are of the same sign, and each term is always greater than some given finite quantity, the series is *divergent*.

Let each of the terms be greater than a given quantity k . Then, the sum of n terms $> nk$.

Now, if n be indefinitely increased, nk is greater than any assignable quantity.

Hence there is no limit to the sum of the series when n is infinite, and therefore the series is divergent.

196. *If the term of an infinite series be alternately positive and negative, and the terms continually decrease, the series is convergent.*

Let $a - b + c - d + e - f + \&c.$ be the series, where the quantities $a, b, c, d, \&c.$, continually decrease.

We then have $\Sigma = (a - b) + (c - d) + (e - f) + \&c., \dots$ (1),
and $\Sigma = a - (b - c) - (d - e) - \&c., \dots$ (2).

From (1), since $a > b, c > d, \&c.$, we learn that $\Sigma > a - b$.

And from (2), since $b > c, d > e, \&c.$, we learn that $\Sigma < a$.

Hence Σ lies between a and $a - b$, and therefore is finite; and the series is convergent.

197. *When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically less than some proper fraction, the series is convergent.*

Let $a_1 + a_2 + a_3 + \&c.$ be the portion of the infinite series, after a finite number of terms, in which the ratio of each term to the preceding is numerically less than a given proper fraction k .

Then we have $a_2 < ka_1, a_3 < ka_2, \&c.$

Hence $\Sigma < a_1 + ka_1 + k^2a_1 + \&c. < \frac{a_1}{1 - k}$.

This portion therefore of the infinite series has a finite limit. Now the sum of the finite number of terms before the terms $a_1, a_2, a_3, \&c.$ is necessarily finite. Hence the sum of the whole series is finite, and the series is convergent.

COR. 1. In the same manner it may be shown that

When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically greater than unity, the series is divergent.

COR. 2. So also we may prove that

When after some finite number, the ratio of each term to the preceding is unity, and the terms have all the same sign, the series is divergent.

Case when the Ratio of each Term to the Preceding approaches Unity.

198. *When the ratio of each term to the preceding is less than unity, but still approaches unity, the series may be convergent or divergent.*

199. *To show that the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \text{dc.}$ is convergent when m is greater than unity, and divergent when m is equal to or less than unity.*

$$\begin{aligned} \text{The ratio of the } n\text{th to the } (n-1)\text{th term} &= \frac{1}{n^m} \div \frac{1}{(n-1)^m} \\ &= \left(\frac{n-1}{n}\right)^m. \end{aligned}$$

This ratio is less than unity, but continually approaches unity as n is indefinitely increased.

(1). Let m be less than unity.

$$\text{Now, } S_n = \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \dots + \frac{1}{n^m}.$$

And the least of these terms is $\frac{1}{n^m}$.

$$\text{Hence, } S_n > n \text{ times } \frac{1}{n^m} > n^{1-m}.$$

But since m is less than unity, $1 - m$ is positive, and the value of n^{1-m} increases indefinitely as n is increased.

Hence, as n is indefinitely increased, the sum of n terms becomes greater than any assignable quantity.

The series is therefore *divergent*.

(2.) Let m be equal to unity.

$$\text{Then } \Sigma = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \&c.$$

$$\therefore \Sigma > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) + \&c. \\ > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c.$$

Hence Σ is greater than any assignable quantity, and the series is *divergent*.

(3) Let m be greater than unity.

$$\text{We have } \Sigma = 1 + \left(\frac{1}{2^m} + \frac{1}{3^m}\right) + \left(\frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \frac{1}{7^m}\right) \\ + \left(\frac{1}{8^m} + \frac{1}{9^m} + \dots + \frac{1}{15^m}\right) + \&c.$$

$$\therefore \Sigma < 1 + 2\left(\frac{1}{2^m}\right) + 4\left(\frac{1}{4^m}\right) + 8\left(\frac{1}{8^m}\right) + \&c. \\ < 1 + \frac{1}{2^{m-1}} + \frac{1}{2^{2m-2}} + \frac{1}{2^{3m-3}} + \&c.$$

Now this is a geometric series whose common ratio is $\frac{1}{2^{m-1}}$.

It has therefore a finite sum. Hence the given series has also a finite sum, and is convergent.

200. If two series, $u_0 + u_1 + u_2 + \dots$, $v_0 + v_1 + v_2 + \dots$, be such that the limit of $\frac{u_n}{v_n}$ when n becomes infinite is a finite quantity, then the two series are both convergent or both divergent.

$$\text{Let } \frac{u_0}{v_0} = k_0, \frac{u_1}{v_1} = k_1, \dots, \frac{u_n}{v_n} = k_n, \&c.$$

And let k be the limit to which k_n continually approaches as n is increased indefinitely.

We have

$$u_n + u_{n+1} + u_{n+2} + \dots = k_n v_n + k_{n+1} v_{n+1} + k_{n+2} v_{n+2} + \dots \dots \dots (1).$$

Now, as k is the limit to which k_n continually approaches, all the ratios $k_n, k_{n+1}, \dots$, must lie between $k + h$ and $k - h$ where h may be as small a quantity as we please.

Hence, $k_n v_n + k_{n+1} v_{n+1} + k_{n+2} v_{n+2} + \dots$ lies between $(k + h) (v_n + v_{n+1} + v_{n+2} + \dots)$ and $(k - h) (v_n + v_{n+1} + v_{n+2} + \dots)$,

where h is as small a quantity as we please.

Hence, from (1), $u_n + u_{n+1} + u_{n+2} + \dots$ lies between $(k + h) (v_n + v_{n+1} + v_{n+2} + \dots)$ and $(k - h) (v_n + v_{n+1} + v_{n+2} + \dots)$,

where h is as small as we please.

Hence the series $u_n + u_{n+1} + u_{n+2} + \dots$ will have a finite limit according as the series $v_n + v_{n+1} + v_{n+2} + \dots$ has a finite limit.

The two series, $u_n + u_{n+1} + u_{n+2} + \dots$, and $v_n + v_{n+1} + v_{n+2} + \dots$, are therefore both convergent or both divergent.

And since the finite number of terms preceding u_n and v_n cannot affect this conclusion, it follows that the given series are either both convergent or both divergent.

201. The series $\phi(1) + \phi(2) + \phi(3) + \phi(4) + \&c.$, and

$$\phi(1) + m \phi(m) + m^2 \phi(m^2) + m^3 \phi(m^3) + \&c.,$$

are convergent and divergent together, when $\phi(x)$ is positive, and continually diminishes as x increases, m being a positive integer.

By extending the first series we may write it thus

$$\begin{aligned} & \phi(1) + \phi(2) + \phi(3) + \phi(4) + \&c. \\ &= \phi(1) \\ &+ \{ \phi(2) + \phi(3) + \dots + \phi(m) \} \\ &+ \{ \phi(m+1) + \phi(m+2) + \dots + \phi(m^2) \} \\ &+ \{ \phi(m^2+1) + \phi(m^2+2) + \dots + \phi(m^3) \} + \&c. \dots (1). \end{aligned}$$

But since the quantities $\phi(1)$, $\phi(2)$, $\phi(3)$, &c. are positive, and continually diminish, we have

$$\phi(2) + \phi(3) + \dots + \phi(m) > (m-1)\phi(m), \text{ and} \\ < (m-1)\phi(1),$$

$$\phi(m+1) + \phi(m+2) + \dots + \phi(m^2) > (m^2-m)\phi(m^2), \text{ and} \\ < (m^2-m)\phi(m),$$

$$\phi(m^2+1) + \phi(m^2+2) + \dots + \phi(m^3) > (m^3-m^2)\phi(m^3), \text{ and} \\ < (m^3-m^2)\phi(m^2),$$

$$\text{\&c.}, \quad \text{\&c.}$$

Hence, adding these inequalities, we have, by (1),

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{\&c.} \\ > \phi(1) + (m-1)\{\phi(m) + m\phi(m^2) + m^2\phi(m^3) + \text{\&c.}\}, \\ \text{and } > \phi(1) + (m-1)\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{\&c.}\}.$$

Or, arranging, we have

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{\&c.} \\ > \frac{1}{m}\phi(1) \\ + \frac{m-1}{m}\{\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \text{\&c.}\} \dots (2).$$

$$\text{And } < \phi(1) + (m-1)\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{\&c.}\} \dots (3).$$

From (2), we learn that when the second series is divergent, the first is divergent; and from (3), that the first is convergent when the second is convergent.

Ex. The series $u_1 + u_2 + u_3 + \dots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when the limit is less than, or equal to, unity.

By what has just been proved,

The two series $u_1 + u_2 + u_3 + \dots$, and $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$ are both convergent or both divergent, if the limit, when n is infinite, of the ratio $u_n : \frac{1}{n^n}$ is a finite quantity.

Let k be this limit.

Then, when n is infinite, we have

$$u_n \cdot n^m = k, \text{ or } \log u_n + m \log n = \log k;$$

$$\text{or } \frac{-\log u_n}{\log n} = m - \frac{\log k}{\log n};$$

or, since the limit of $\frac{\log k}{\log n} = 0$, the limit of $\frac{-\log u_n}{\log n} = m$.

Now, by Art. 189, the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \dots$ is convergent when m is greater than unity, and divergent when m is less than, or equal to, unity.

Hence, since m is the limit of $\frac{-\log u_n}{\log n}$, and in that case the convergency or divergency of the series $u_1 + u_2 + u_3 + \dots$ depends upon that of the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \&c.$, it follows that the series $u_1 + u_2 + u_3 + \dots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when equal to, or less than unity.

Ex. XLI.

Test the convergency or divergency of the following series:—

$$1. \frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \dots$$

$$2. \frac{1}{\lfloor 2 \rfloor} + \frac{2}{\lfloor 3 \rfloor} + \frac{3}{\lfloor 4 \rfloor} + \dots$$

$$3. \frac{1}{\lfloor 2 \rfloor} + \frac{1 \cdot 3}{\lfloor 3 \rfloor} + \frac{1 \cdot 3 \cdot 5}{\lfloor 4 \rfloor} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{\lfloor 5 \rfloor} + \dots$$

$$4. \frac{a}{\lfloor a+1 \rfloor} + \frac{2a^2}{\lfloor a+2 \rfloor} + \frac{3a^3}{\lfloor a+3 \rfloor} + \dots$$

$$5. \frac{a}{a+b} + \frac{a^2}{(a+b)(a+2b)} + \frac{a^3}{(a+b)(a+2b)(a+3b)} + \dots$$

$$6. \frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \dots$$

$$7. 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

$$8. 1 + \frac{2n(2n-1)}{1 \cdot 2} + \frac{2n(2n-1)(2n-2)(2n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

$$9. \frac{x}{2} + \frac{2^2 x^2}{2^2 + 1} + \frac{3^2 x^3}{3^2 + 1} + \dots + \frac{n^2 x^n}{n^2 + 1} + \dots$$

$$10. 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \dots$$

11. Show that the series whose n th term is $\phi(n)x^n$ is convergent or divergent according as the limit of $\frac{\phi(n)}{\phi(n-1)}$, when n is infinite, is less or greater than unity.

• If the limit be negative, or unity, what is the test you apply?

12. Show that the $\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$ is convergent, and that the error in taking the first n terms for the series is less than $\frac{1}{(n-1)(n-1)}$.

Recurring Series.

202. The series $a_0 + a_1x + a_2x^2 + \dots$ is called a **recurring series**, when the relation existing between every set of (r) consecutive terms can be expressed by the equation

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \dots + p_{r-1}a_{n-r+1} = 0;$$

where $p_1, p_2, p_3, \dots, p_{r-1}$ are constant quantities, and where n may have any integral value.

It is usual to call $1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}$ the **scale of relation**.

A simple case is an ordinary geometric series.

Thus, in the series $1 + 2x + 4x^2 + 8x^3 + \dots$, we have

$$a_0 = 1, a_1 = 2, a_2 = 4, \text{ \&c.}, a_n = 2^n, \text{ \&c.}$$

$$\text{Hence } \frac{a_n}{a_{n-1}} = \frac{2^n}{2^{n-1}} = 2, \text{ or } a_n - 2a_{n-1} = 0.$$

This equation expresses generally the relation existing between any *two* consecutive terms.

So we have $a_1 - 2a_0 = 0$, $a_2 - 2a_1 = 0$, $a_3 - 2a_2 = 0$, &c. And the *scale of relation* of the series is $1 - 2x$.

The following examples will illustrate the method of finding the scale of relation of a given series.

Ex. 1. Given the series $2 + 2x - 4x^2 - 22x^3 - 46x^4 - \text{ \&c.}$, find the scale of relation.

We can see that the given series is not an ordinary G. P., and therefore its scale of relation will consist of more than *two* terms.

Assume it to be $1 + p_1x + p_2x^2$.

$$\begin{aligned} \text{Then we have} \quad & -4 + 2p_1 + 2p_2 = 0 \dots\dots\dots(1), \\ & -22 - 4p_1 + 2p_2 = 0 \dots\dots\dots(2), \\ & -46 - 22p_1 - 4p_2 = 0 \dots\dots\dots(3). \end{aligned}$$

From (1) and (2) we get $p_1 = -3$, $p_2 = 5$, and these values also satisfy (3). Hence we conclude that our *assumption* is correct, and that the scale of relation is $1 - 3x + 5x^2$.

We *might have found* that the value of p_1 and p_2 in (1) and (2) would not satisfy (3), and we must then have made another assumption.

For instance, we should have assumed the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

In that case we must have had given us another term in the series; for we should have required *three* equations to determine the values of p_1 , p_2 , p_3 , and a *fourth* to verify our results. In the next example we shall assume the scale of relation to consist of *four* terms, as on trial it would be found $1 + p_1x + p_2x^2$ would not answer the conditions.

Ex. 2. Find the scale of relation of the series

$$1 + 10x + 34x^2 + 67x^3 + 115x^4 + 247x^5 + 716x^6 + \&c.$$

Assume the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

$$\text{Then } 67 + 34p_1 + 10p_2 + p_3 = 0.$$

$$115 + 67p_1 + 34p_2 + 10p_3 = 0.$$

$$247 + 115p_1 + 67p_2 + 34p_3 = 0.$$

$$716 + 247p_1 + 115p_2 + 67p_3 = 0.$$

From the first three equations we find $p_1 = -3$, $p_2 = 4$, $p_3 = -5$; and these values satisfy the *fourth* equation.

Hence the scale of relation is $1 - 3x + 4x^2 - 5x^3$.

203. To find the sum of a recurring series.

Let $a_0 + a_1x + a_2x^2 + \dots$ be the series, whose scale of relation is $1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}$.

We have then

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \dots + p_{r-1}a_{n-r+1} = 0 \dots (1),$$

where n has any positive integral value.

Let $\Sigma = a_0 + a_1x + a_2x^2 + \dots + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \dots$

Multiplying each side of the equation by $1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}$, we have

$$\begin{aligned} (1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}) \Sigma \\ = a_0 + a_1x + a_2x^2 + \dots + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \dots \\ + p_1a_0x + p_1a_1x^2 + \dots + p_1a_{r-2}x^{r-2} + p_1a_{r-1}x^{r-1} + \dots \\ + p_2a_0x^2 + \dots + p_2a_{r-2}x^{r-2} + p_2a_{r-1}x^{r-1} + \dots \\ + \dots \dots \dots \\ + p_{r-2}a_0x^{r-2} + p_{r-2}a_1x^{r-1} + \dots \\ + p_{r-1}a_0x^{r-1} + \dots \end{aligned}$$

Collecting the coefficients of like powers of x , and remembering that the coefficient of x^{r-1} , viz.,

$$a_{r-1} + p_1a_{r-2} + p_2a_{r-3} + \dots + p_{r-2}a_1 + p_{r-1}a_0,$$

and all succeeding coefficients are each equal to zero, as we learn from (1) above, we get

$$\begin{aligned}
 & (1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}) \Sigma \\
 &= a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \dots + (a_{r-2} + p_1a_{r-3} + \dots + p_{r-2}a_0)x^{r-2} \\
 \therefore \Sigma &= \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \dots + (a_{r-2} + p_1a_{r-3} + \dots + p_{r-2}a_0)x^{r-2}}{1 + p_1x + p_2x^2 + \dots + p_{r-1}x^{r-1}} \quad (A).
 \end{aligned}$$

Cor. When $r = 3$, or the scale of relation is $1 + p_1x + p_2x^2$, we have

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x}{1 + p_1x + p_2x^2} \dots\dots\dots (2).$$

And so, when $r = 4$, or the scale of relation is $1 + p_1x + p_2x^2 + p_3x^3$,

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2}{1 + p_1x + p_2x^2 + p_3x^3} \dots\dots\dots (3).$$

Ex. 1. Sum the infinite series $2 + 2x - 4x^2 - 22x^3 - 46x^4 - \dots$.

By Art. 90, Ex. 1, the scale of relation is $1 - 3x + 5x^2$.

Hence from (3) above, remembering that $a_0, a_1, \dots$, represent the coefficients of the given series, and that p_1, p_2 are the coefficients of the powers of x in the scale of relation, we have

$$\Sigma = \frac{2 + \{2 + (-3)2\}x}{1 - 3x + 5x^2} = \frac{2 - 4x}{1 - 3x + 5x^2}.$$

Ex. 2. Find the sum of the infinite series

$$1 + 10x + 34x^2 + 67x^3 + 115x^4 + 247x^5 + 716x^6 + \&c$$

We found the scale of relation to be $1 - 3x + 4x^2 - 5x^3$.

Hence, by (3) above,

$$\begin{aligned}\Sigma &= \frac{1 + \{10 - 3(1)\}x + \{34 - 3(10) + 4(1)\}x^2}{1 - 3x + 4x^2 - 5x^3} \\ &= \frac{1 + 7x + 8x^2}{1 - 3x + 4x^2 - 5x^3}.\end{aligned}$$

Since, by the process of long division, or otherwise, the expression found in (A) above will give the original series, we give to the value of Σ there found the name of the *generating function* of the series.

204. *When the scale of relation can be broken up into simple factors, it is easy to find the general term of the series and the sum to n terms.*

Let the scale of relation be capable of being broken up into the $(r - 1)$ factors, $(1 - b_1x)$, $(1 - b_2x)$, &c.

Then the generating function can be resolved into partial fractions having these factors for denominators.

Thus we have

$$\begin{aligned}\Sigma &= \frac{P_1}{1 - b_1x} + \frac{P_2}{1 - b_2x} + \dots + \frac{P_{r-1}}{1 - b_{r-1}x} \\ &= P_1(1 + b_1x + b_1^2x^2 + \dots + b_1^{n-1}x^{n-1} + \dots) \\ &\quad + P_2(1 + b_2x + b_2^2x^2 + \dots + b_2^{n-1}x^{n-1} + \dots) \\ &\quad + \&c. \\ &\quad + P_{r-1}(1 + b_{r-1}x + b_{r-1}^2x^2 + \dots + b_{r-1}^{n-1}x^{n-1} + \dots)\end{aligned}$$

We thus find the n th term of the series to be

$$(P_1b_1^{n-1} + P_2b_2^{n-1} + \dots + P_{r-1}b_{r-1}^{n-1})x^{n-1}.$$

Also, the sum to n terms,

$$\begin{aligned}&= P_1 \cdot \frac{1 - (b_1x)^n}{1 - b_1x} + P_2 \cdot \frac{1 - (b_2x)^n}{1 - b_2x} + \dots \\ &\quad + P_{r-1} \cdot \frac{1 - (b_{r-1}x)^n}{1 - b_{r-1}x}.\end{aligned}$$

Ex. Find the sum to n terms and to infinity of the series
 $8 - 29x + 257x^2 - 1691x^3 + 12053x^4 - \&c.$

The scale of relation is easily found to be $1 + 5x - 14x^2$.

$$\therefore \Sigma = \frac{8 + \{-29 + 5(8)\}x}{1 + 5x - 14x^2} = \frac{8 + 11x}{(1 - 2x)(1 + 7x)}.$$

Resolving into partial fractions, we have

$$\begin{aligned}\Sigma &= \frac{3}{1 - 2x} + \frac{5}{1 + 7x} \\ &= 3 \{1 + 2x + (2x)^2 + \dots + (2x)^{n-1} + \dots\} \\ &\quad + 5 \{1 - 7x + (7x)^2 + \dots + (-7x)^{n-1} + \dots\}.\end{aligned}$$

Therefore sum to n terms, or,

$$S = 3 \cdot \frac{1 - (2x)^n}{1 - 2x} + 5 \cdot \frac{1 - (-7x)^n}{1 + 7x}.$$

When the given recurring term does not contain the powers of x , but is of the form $a_0 + a_1x + a_2x^2 + \dots$, we may first find the scale of relation and the sum of the series $a_0 + a_1x + a_2x^2 + \dots$, and put $x = 1$ in the result.

Summation of Series.

205. Certain series may be summed by particular artifices.

In the series $u_1 + u_2 + \dots + u_n$, suppose u_n to be broken up into the difference of two terms, one of which is the same function of $(n - 1)$ as the other is of n .

I. Thus, suppose $u_n = U_n - U_{n-1}$.

Then, putting for n all values from 1 to n , we have

$$\begin{aligned}S &= (U_1 - U_0) + (U_2 - U_1) + \dots + (U_n - U_{n-1}) \\ &= U_n - U_0.\end{aligned}$$

II. Suppose $u_n = U_{n-1} - U_n$.

We have, similarly,

$$S = (U_0 - U_1) + (U_1 - U_2) + \dots + (U_{n-1} - U_n) \\ = U_0 - U_n$$

Ex. 1. Sum the series

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1).$$

We have

$$u_n = n(n+1) = \frac{1}{3} n(n+1) \{(n+2) - (n-1)\} \\ = \frac{1}{3} n(n+1)(n+2) - \frac{1}{3} (n-1)n(n+1) \\ = U_n - U_{n-1}$$

where $U_n = \frac{1}{3} n(n+1)(n+2),$

and therefore $U_0 = \frac{1}{3} (0)(1)(2) = 0.$

Hence, S or $U_n - U_0 = \frac{1}{3} n(n+1)(n+2) - 0.$

$$\therefore S = \frac{1}{3} n(n+1)(n+2).$$

Ex. 2. Sum the series

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)}.$$

Here $u_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \cdot \frac{(2n+1) - (2n-1)}{(2n-1)(2n+1)} \\ = \frac{1}{2} \cdot \frac{1}{2n-1} - \frac{1}{2} \cdot \frac{1}{2n+1} = U_{n-1} - U_n$

where $U_n = \frac{1}{2} \cdot \frac{1}{2n+1},$ and $\therefore U_0 = \frac{1}{2} \cdot \frac{1}{2(0)+1} = \frac{1}{2}.$

Hence, by II. above

$$S = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1} = \frac{n}{2n+1}.$$

206. When the n th term of a series is of the form

$$u_n = (a + b)(a + n + 1 + b) \dots (a + n + m - 1 + b),$$

where a, b are constants, and m the number of factors, then $S = U_n - U_0$, where

$$U_n = \frac{(an + b)(a \cdot \overline{n+1} + b) \dots (a \cdot \overline{n+m} + b)}{(m+1)a};$$

that is, U_n is found by taking one factor more, and dividing by as many times the common difference as there are then factors.

$$\begin{aligned} \text{For } u_n &= \frac{u_n}{(m+1)a} \left\{ (a \cdot \overline{n+m} + b) - (a \cdot \overline{n-1} + b) \right\} \\ &= \frac{u_n(a \cdot \overline{n+m} + b)}{(m+1)a} - \frac{(a \cdot \overline{n-1} + b)u_n}{(m+1)a} \\ &= \frac{(an+b)(a \cdot \overline{n+1} + b) \dots (a \cdot \overline{n+m} + b)}{(m+1)a} \\ &\quad - \frac{(a \cdot \overline{n-1} + b)(an+b) \dots (a \cdot \overline{n+m-1} + b)}{(m+1)a} \\ &= U_n - U_{n-1} \end{aligned}$$

$$\text{where } U_n = \frac{(an+b)(a \cdot \overline{n+1} + b) \dots (a \cdot \overline{n+m} + b)}{(m+1)a} \dots (I).$$

Hence, by Art. 205, $S = U_n - U_0$, where U_n has the value in (I).

Ex. 1. Sum the series

$$3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \dots + (2n+1)(2n+3)(2n+5).$$

We have $u_n = (2n+1)(2n+3)(2n+5)$.

Here the common difference is 2, and with one more factor, the number of factors will be 4.

$$\text{Hence } U_n = \frac{(2n+1)(2n+3)(2n+5)(2n+7)}{4(2)}$$

$$= \frac{1}{8}(2n+1)(2n+3)(2n+5)(2n+7).$$

$$\text{Therefore } U_0 = \frac{1}{8}(1)(3)(5)(7) = \frac{105}{8}.$$

$$\therefore S = \frac{1}{8} \{ (2n+1)(2n+3)(2n+5)(2n+7) - 105 \}$$

$$= 2n^4 + 16n^3 + 43n^2 + 44n.$$

When u_n consists of the *sum of several terms* of the form in Art. 206, it is evident that U_n must also consist of the sum of an equal number of terms, each obtained according to the law there proved. The next example is an illustration.

Ex. 2. Sum the series $1^3 + 2^3 + 3^3 + \dots + n^3$.

$$\text{We have } u_n = n^3 = n \{ (n-1)(n+1) + 1 \}$$

$$= (n-1)n(n+1) + n$$

$$\text{Hence } U_n = \frac{(n-1)n(n+1)(n+2)}{4(1)} + \frac{n(n+1)}{2(1)}$$

$$= \frac{1}{4} n^2 (n+1)^2.$$

Therefore $U_0 = 0$.

$$\text{Hence } S \text{ or } U_n - U_0 = \frac{1}{4} n^2 (n+1)^2 = \left\{ \frac{1}{2} n(n+1) \right\}^2.$$

207. When the n th term of a series is of the form

$$u_n = \frac{1}{(an+b)(a.n+1+b) \dots (a.n+m-1+b)},$$

where a, b are constants, and m the number of factors, then $S = U_0 - U_m$, where

$$U_n = \frac{1}{(m-1)a.(a.n+1+b)(a.n+2.b) \dots (a.n+m-1.b)};$$

that is, where U_n is found by omitting the first factor, and dividing by as many times the common difference as there are factors remaining.

For u_n

$$= \frac{1}{(m-1)a} \cdot \frac{(a.n+m-1+b) - (an+b)}{(an+b)(a.n+1+b) \dots (a.n+m-1+b)}$$

$$= \frac{1}{(m-1)a} \left\{ \frac{1}{(am+b)(a \cdot n+1+b)} \dots \frac{1}{(a \cdot n+m-2+b)} \dots \frac{1}{(a \cdot n+m-1+b)} \right\} \\ = U_{n-1} - U_n$$

$$\text{where } U_n = \frac{1}{(m-1)a \cdot (a \cdot n+1+b) \dots (a \cdot n+m-1+b)} \dots \dots \dots (1).$$

Hence, by Art. 93, $S = U_0 - U_n$ where U_n has the value in (1).

Ex. 1. Sum the series $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots + \frac{1}{n(n+1)(n+2)}$.

$$\text{Here } u_n = \frac{1}{n(n+1)(n+2)}.$$

$$\text{Therefore } U_n = \frac{1}{2(1)(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)}.$$

$$\text{Hence also, } U_0 = \frac{1}{2(1)(2)} = \frac{1}{4}.$$

$$\text{Therefore } U_0 - U_n \text{ or } S = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

Hence also, since the limit of $\frac{1}{2(n+1)(n+2)}$ is 0, when $n = \infty$, we have $\Sigma = \frac{1}{4}$.

Ex. 2. Sum the series $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots + \frac{2n-1}{n(n+1)(n+2)}$.

$$\begin{aligned}\text{Here } u_n &= \frac{2n-1}{n(n+1)(n+2)} \\ &= \frac{2}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}.\end{aligned}$$

$$\begin{aligned}\text{Therefore } U_n &= \frac{2}{1(1)(n+2)} - \frac{1}{2(1)(n+1)(n+2)} \\ &= \frac{2}{n+2} - \frac{1}{2(n+1)(n+2)} \\ &= \frac{4n+3}{2(n+1)(n+2)}.\end{aligned}$$

$$\text{Hence } U_0 = \frac{3}{2(1)(2)} = \frac{3}{4}.$$

$$\text{Therefore } U_0 - U_n, \text{ or } S = \frac{3}{4} - \frac{4n+3}{2(n+1)(n+2)}.$$

$$\text{Hence also, since } \frac{4n+3}{2(n+1)(n+2)} = \frac{4+\frac{3}{n}}{2(n+1)(1+\frac{2}{n})}$$

$$= 0, \text{ when } n = \infty, \text{ we have } S = \frac{3}{4}.$$

Figurate Numbers.

208. We give the name **Figurate Numbers** to those of the following series, where the n th term of any series is the sum of n terms of the preceding series:—

1st order, 1, 1, 1, 1, &c.

2nd „ 1, 2, 3, 4, &c.

3rd „ 1, 3, 6, 10, &c.

&c.

For the first order, we have $S_n = n$.

And for the second, $S_n = \frac{1}{2}n(n+1)$.

This is the n th term of the 3rd order.

Hence, to find the sum of n terms of the third order, we have

$$u_n = \frac{1}{2}n(n+1).$$

Therefore (Art. 206),

$$U_n = \frac{1}{2} \cdot \frac{n(n+1)(n+2)}{3(1)} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

And therefore $U_0 = 0$.

$$\text{Hence } U_n - U_0 \text{ or } S_n = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

To find S_n for the r th order,

$$\text{Assume that for the } p\text{th order, } S_n = \frac{n(n+1) \dots (n+p-1)}{1 \cdot 2 \dots p}.$$

$$\text{Then for the } (p+1)\text{th order, } u_n = \frac{n(n+1) \dots (n+p-1)}{1 \cdot 2 \dots p}.$$

Therefore

$$U_n = \frac{n(n+1) \dots (n+p)}{(p+1) \cdot 1 \cdot 2 \dots p} = \frac{n(n+1) \dots (n+p+1-1)}{1 \cdot 2 \dots p+1}.$$

$$\text{and } U_0 = 0.$$

$$\text{Therefore } U_n - U_0 \text{ or } S_n = \frac{n(n+1) \dots (n+p+1-1)}{1 \cdot 2 \dots p+1}.$$

Hence, by assuming S_n to be formed according to a certain law for the p th order, we have proved it to be formed according to the *same law* for the $(p+1)$ th order. Now, we know it is true for the 2nd and 3rd orders. It is therefore true for the 4th, and therefore for the 5th, and so generally true.

Hence for the r th order,

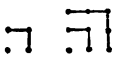
$$S_n = \frac{n(n+1) \dots (n+r-1)}{1 \cdot 2 \dots r}.$$

Polygonal Numbers.

209. These numbers are called *linear, triangular, square, pentagonal*, and generally *polygonal*, from the fact that if a dot be taken to represent unity, they may all be represented in the form of the corresponding polygons. Thus—

Linear Numbers, ... , &c., or 1, 2, 3, 4, &c.

Triangular „ . . \ \ , &c., or 1, 3, 6, 10, &c.

Square „ . .  , &c., or 1, 4, 9, 16, &c.

Pentagonal „ . .  , &c., or 1, 5, 12, 22, &c.

The *first order* of polygonal numbers is the series in which each term is *unity*.

And hence, by observing the above diagrams, it is easy to see that the several orders may be arranged as follows:—

1st order,	1,	1	1	1	&c.
2nd „	1,	1 + 1,	1 + 1 + 1,	1 + 1 + 1 + 1,	&c.
3rd „	1,	1 + 2,	1 + 2 + 3,	1 + 2 + 3 + 4,	&c.
4th „	1,	1 + 3,	1 + 3 + 5,	1 + 3 + 5 + 7,	&c.
5th „	1,	1 + 4,	1 + 4 + 7,	1 + 4 + 7 + 10,	&c.

The law is evident. The n th number of each order after the first is the sum of n terms of an A. P., whose first term is unity; and the common difference in the r th order is $r - 2$.

Hence the n th term of the r th order

$$= 1 + (1 + r - 2) + (1 + 2 \cdot r - 2) + \&c., \text{ to } n \text{ terms}$$

$$= \left\{ 2 + n - 1 \cdot (r - 2) \right\} \frac{n}{2}$$

$$= n + \frac{1}{2} n (n - 1) (r - 2).$$

Hence, for the r th order, we have u_n

$$= n + \frac{1}{2} (r - 2) \cdot (n - 1)n.$$

$$\text{Hence, } U_n = \frac{n(n+1)}{2} + \frac{1}{2} (r - 2) \cdot \frac{(n-1)n(n+1)}{3},$$

and $\therefore U_0 = 0$.

Hence (Art. 205) for the r th order.

$$S_n = \frac{n}{6} (n + 1) \{ (r - 2) (n - 1) + 3 \}.$$

Cor. Let $r = 3, 4, 5$, successively, then,
 For triangular numbers, $S_n = \frac{1}{6} n(n+1)(n+2)$.
 „ square „ $S_n = \frac{1}{6} n(n+1)(2n+1)$.
 „ pentagonal „ $S_n = \frac{1}{6} n(n+1)(3n-3+3)$
 $= \frac{1}{2} n^2(n+1)$.

210. To find the number of cannon balls in a pyramidal heap.

(1.) When the base is an *equilateral triangle*.

Let n be the number of balls in a side of the base.

Then, number of balls in lowest layer

$$= 1 + 2 + 3 + \dots + n = \frac{1}{2} n(n+1),$$

the n th triangular number.

Hence, by Art. 205, $S_n = \frac{1}{6} n(n+1)(n+2)$.

(2.) When the base is a *square*.

We have, similarly, $S_n = \frac{1}{6} n(n+1)(2n+1)$.

(3.) When the pile is *deficient*, the first m courses being removed.

Here, if S_n = the number in the *complete* pile,

and S_m = the number in the removed courses,

$S_n - S_m$ = the number in the incomplete pile.

211. To find the number of balls when the base is *rectangular*, and not square.

The pile will now terminate in a single row at the top.

Let l be the length of the top row.

Then, the n th row, reckoned from the top, will contain $l + n - 1$ balls, and its breadth will be n .

Then, number of balls in the lowest or n th layer

$$= (l + n - 1) n$$

Hence $u_n = (l + n - 1) n = l \cdot n + (n - 1) n$.

Therefore (Art. 206)

$$U_n = l \cdot \frac{n(n+1)}{2} + \frac{(n-1)n(n+1)}{3}; \text{ or, since } U_0 = 0,$$

T

$$\begin{aligned}
 S_n &= l \cdot \frac{n(n+1)}{2} + \frac{(n-1)n(n+1)}{3} \\
 &= \frac{1}{6} n(n+1) \{3l + 2(n-1)\} \\
 &= \frac{1}{6} n(n+1) \{3(l+n-1) - n + 1\} \\
 &= \frac{1}{6} n(n+1) (3l_1 - n + 1),
 \end{aligned}$$

when l_1 is the length, and n the breadth, of the lowest row.

Method of Differences.

212. Suppose we have to find the sum of the series
3, 6, 11, 18, 27, 38, &c.

Taking the differences of each two consecutive terms, and then the differences of each two of these differences, and so on, until the differences become a series of identical terms, we have :

Series, ... 3, 6, 11, 18, 27, 38, &c.
1st order of differences, . . . 3, 5, 7, 9, 11, &c.
2nd order of differences, . . . 2, 2, 2, 2, &c.

It will be seen that *the n th term of any of these series is the sum of its 1st term, and the first $(n-1)$ terms of the succeeding series*..... (A).

Thus, in the 1st order,

$$u_n = 3 + 2(n-1).$$

$$\begin{aligned}
 \text{Therefore, by Art. 206, } U_n &= 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)n}{2 \cdot 1} \\
 &= 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)n}{1 \cdot 2}; \text{ or, since } U_0 = 0
 \end{aligned}$$

$$S_n = 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)n}{1 \cdot 2}.$$

$$\therefore S_{n-1} = 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)(n-1)}{1 \cdot 2}.$$

Hence, by (A) above, in the given series,

$$u_n = 3 + 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)(n-1)}{1 \cdot 2}.$$

$$\therefore U_n = 3 \cdot \frac{n}{1} + 3 \cdot \frac{(n-1)n}{1 \cdot 2} + 2 \cdot \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3};$$

or, since $U_0 = 0$,

$$\begin{aligned} S_n &= 3n + 3 \cdot \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, \\ &= \frac{n}{6}(n^2 + 3n + 13). \end{aligned}$$

We shall in the next article give the general theorem.

213. If u_1, u_2, u_3 , &c. be a series, and d_1, d_2, d_3 , &c. the first terms of the successive order of differences, the differences in the r th order being identical, then we have

$$\begin{aligned} S_n &= nu_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \dots \\ &\quad + \frac{n(n-1) \dots (n-r)}{r+1} d_r \end{aligned}$$

In the $(r-1)$ th order, we have $u_n = d_{r-1} + (n-1)d_r$

Therefore $U_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r$; and therefore, since

$$U_0 = 0, S_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r$$

Therefore $S_{n-1} = \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r$

In the $(r-2)$ th order we therefore have

$$u_n = d_{r-2} + \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r$$

Therefore $U_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r$;

and therefore, since $U_0 = 0$,

$$S_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r$$

We may easily show therefore by induction that—

For the $(r - p)$ th order

$$S_n = \frac{n}{1} d_{r-p} + \frac{(n-1)n}{1.2} d_{r-p+1} + \dots \\ + \frac{(n-p)(n-p+1) \dots (n-1)n}{1.2 \dots (p+1)} d_r$$

Put $p = r$, then for the series *preceding* the 1st order of differences, that is, for the given series, since $d_0 = u_1$, we have

$$S_n = \frac{n}{1} u_1 + \frac{(n-1)n}{1.2} d_1 + \frac{(n-2)(n-1)n}{1.2.3} d_2 + \dots \\ + \frac{(n-r) \dots (n-1)n}{1.2 \dots (r+1)} d_r \\ = nu_1 + \frac{n(n-1)}{1.2} d_1 + \frac{n(n-1)(n-2)}{1.2.3} d_2 + \dots \\ + \frac{n(n-1) \dots (n-r)}{r+1} d_r$$

Ex. XLII.

Find the generating functions of the following six series :

1. $1 + 8x + 22x^2 + 50x^3 + 106x^4 + \dots$
2. $1 + 3x + x^2 - 5x^3 - 7x^4 - \dots$
3. $2 + 13x + 38x^2 + 61x^3 - 22x^4 - \dots$
4. $3 + 11x + 57x^2 + 309x^3 + 1683x^4 + \dots$
5. $9 - 11x - 52x^2 + 129x^3 + 235x^4 - \dots$
6. $6 + 16x + 25x^2 + 17x^3 - 33x^4 - 142x^5 - 277x^6 - \dots$

Sum to n terms, and to infinity, the following eight series :

7. $3 - 13x + 69x^2 - 337x^3 + 170x^4 - \dots$
8. $6 - 33x + 249x^2 - 1707x^3 + 12021x^4 - \dots$
9. $1 - 2x + 7x^2 - 8x^3 + 37x^4 - 14x^5 + 211x^6 + \dots$
10. $1 - 7x + 22x^2 - 70x^3 + 211x^4 - 637x^5 + 1912x^6 - \dots$
11. $2 - 11x + 54x^2 - 271x^3 + 1354x^4 - \dots$

12. $1 + 4x + 10x^2 + 34x^3 - 26x^4 + \dots$
13. $3 + 14x + 36x^2 + 69x^3 + 113x^4 + 168x^5 + 234x^6 + \dots$
14. $4 + 29x + 131x^2 + 491x^3 + 1679x^4 + 5459x^5 + 17231x^6 + \dots$
15. Find the scale of relation of the recurring series
 $1^2 + 2^2 + 3^2 + 4^2 + \dots$
16. Find the generating function of the series
 $a + (a + b)x + (a + 2b)x^2 + (a + 3b)x^3 + \dots$
17. If $a_0, a_1, a_2, \dots$ form an A. P., and $b_0, b_1, b_2, \dots$ form a G. P., show that the series
 $\frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots$ is recurring.
18. If $1 + p_1x + p_2x^2$ is the scale of relation of the series
 $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, show that
 $a_n(a_n^2 - a_{n-1}a_{n+1}) + a_{n-1}(a_{n-1}a_{n+2} - a_na_{n+1})$
 $+ a_{n-2}(a_{n-1}^2 - a_na_{n+2}) = 0.$
- Sum to n terms the following ten series:
19. $1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \dots$
20. $(x+1)(x+2)(x+3) + (x+2)(x+3)(x+4) + \dots$
21. $\frac{1}{2 \cdot 4 \cdot 6} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{4 \cdot 6 \cdot 8} + \dots$
22. $\frac{4}{2 \cdot 3 \cdot 4} + \frac{7}{3 \cdot 4 \cdot 5} + \frac{10}{4 \cdot 5 \cdot 6} + \dots$
23. $\frac{1^3 + 1}{2} + \frac{2^3 + 1}{3} + \frac{3^3 + 1}{4} + \dots$
24. $\frac{1}{(a+b)(a+2b)(a+3b)}$
 $+ \frac{1}{(a+2b)(a+3b)(a+4b)} + \dots$
25. $1^2 + 3^2 + 5^2 + \dots$
26. $1^3 + 3^3 + 5^3 + \dots$

$$27. \frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

$$28. \frac{1}{3} + \frac{12}{4} + \dots + \frac{3n^2 + 2n - 4}{n + 2}.$$

$$29. \frac{1}{m} + \frac{m-n}{m(m-1)} + \frac{(m-n)(m-n-1)}{m(m-1)(m-2)} + \dots \text{to } m-n+1 \text{ terms.}$$

$$30. \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{4^2} + \dots \text{to infinity.}$$

31. Find the number of shot in a triangular pile, each side of whose base contains 10 balls.

32. There are 12 balls in the length, and 8 in the breadth of the base of a rectangular pile, and it terminates in a single row. Find the number of balls.

33. Find the number of balls in a rectangular pile when m, n represent respectively the number in the length and breadth of the base, and there are p layers.

34. If $x + \{(1-x)^2 - cx\}$ be expanded in a series of ascending powers of x , show that the coefficient of x^n is

$$r \left\{ 1 + \frac{r^2 - 1}{2 \cdot 3} c + \frac{(r^2 - 1)(r^2 - 4)}{2 \cdot 3 \cdot 4 \cdot 5} c^2 + \frac{(r^2 - 1)(r^2 - 4)(r^2 - 9)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} c^3 + \dots \right\}.$$

35. Show that

$$\frac{1}{m} - n \cdot \frac{1}{m+p} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{m+2p} - \dots + (-1)^n \cdot \frac{1}{m+np} \\ = \frac{np^n}{m(m+p) \dots (m+np)}.$$

36. If p_r be the coefficient of the $(r+1)$ th term of $(1+x)^n$, where n is a positive integer, then

$$\frac{p_1}{p_0} + \frac{2p_2}{p_1} + \frac{3p_3}{p_2} + \dots + \frac{np_n}{p_{n-1}} = \frac{1}{2} n(n+1),$$

$$\text{and } (p_0 + p_1)(p_1 + p_2)(p_2 + p_3) \dots (p_{n-1} + p_n) \\ = \frac{p_1 p_2 p_3 \dots p_n}{1 \cdot 2 \cdot 3 \dots n} (n+1)^n.$$

CHAPTER XXIV.

Scales of Notation.

214. An ordinary number as 6437 may be thus written :

$$6 \cdot 10^3 + 4 \cdot 10^2 + 3 \cdot 10 + 7.$$

We call 10 the radix of the decimal system of notation.

And generally, if r be the radix of any other scale, and $p_0, p_1, p_2, \dots, p_{n-1}$, the n digits commencing with the one at the right hand, we may express a number N thus :

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots + p_2r^2 + p_1r + p_0.$$

It is easy to see that in order to express all numbers in any given scale, it is necessary and sufficient to have, besides zero, one digit less than the radix of the scale.

Thus, in ordinary notation, we have 9 digits.

Hence also, for the duodenary scale we shall require characters to represent *ten* and *eleven*. It is usual to represent them by $\text{\textit{t}}$ and $\text{\textit{e}}$ respectively.

Ex. Express 1420 (senary scale), and $1\text{\textit{t}}e3$ (duodenary scale), as common numbers.

$$1420 \text{ (senary scale)} = 1 \cdot 6^3 + 4 \cdot 6^2 + 2 \cdot 6 + 0 = 372.$$

$$1\text{\textit{t}}e3 \text{ (duodenary scale)} = 1 \cdot 12^3 + 10 \cdot 12^2 + 11 \cdot 12 + 3 = 3303.$$

215. To transform an integer from any scale of notation to any other scale.

Let N be the number, and r the radix of the scale in which we are to express it.

We shall therefore have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots + p_2r^2 + p_1r + p_0 \dots (1),$$

where p_0, p_1, p_2 , &c., are the digits whose values are required.

From (1), dividing each side by r , we have

$$\text{Quotient} = p_{n-1}r^{n-2} + p_{n-2}r^{n-3} + \dots + p_2r + p_1 \dots (2),$$

and remainder = p_0

We obtain then the first or right hand digit by dividing the given number by the proposed radix, and taking the remainder.

Again, dividing each side of (2) by r , we get—

Quotient = $p_{n-1}r^{n-2} + p_{n-2}r^{n-3} + \dots + p_1 \dots (3)$,
and Remainder = p_0 .

Hence, we obtain the second digit by dividing the first quotient by the radix, and taking the remainder.

And so we may proceed until we have found all the digits. We have therefore the following rule:—

RULE.—Divide the given number by the proposed radix, then the quotient so obtained again by the radix, and so on. Then the successive remainders are the respective digits required, commencing with the right hand one.

Ex. 1. Transform the common number 3526 into the ternary and septenary scales.

3	3526	7	3526
3	1175 1	7	503 5
3	391 2	7	71 6
3	130 1	7	10 1
3	43 1	7	1 3
3	14 1		 1
3	4 2		
3	1 1		
	 1		

Hence 3526, in the common scale, is expressed by 11211121 in the ternary scale, and by 13165 in the septenary scale.

Ex. 2. Transpose *35et* from the duodenary scale to the septenary scale.

First method.

Expressing *35et* as a common number, we have

$$35et = 3 \cdot 12^3 + 5 \cdot 12^2 + 11 \cdot 12 + 10 = 6046.$$

Then, as in the last example,

$$\begin{array}{r|l} 7 & 6046 \\ \hline 7 & 863 \dots 5 \\ \hline 7 & 123 \dots 2 \\ \hline 7 & 17 \dots 4 \\ \hline 7 & 2 \dots 3 \\ \hline & \dots 2 \end{array}$$

Therefore, the number required in the septenary scale is 23425.

Second method.

We may proceed to divide the given number, *35et* at once by 7. Thus—

$$\begin{array}{r|l} 7 & 35et \\ \hline 7 & 5e \dots 5 \\ \hline 7 & t3 \dots 2 \\ \hline 7 & 15 \dots 4 \\ \hline 7 & 2 \dots 3 \\ \hline & \dots 2 \end{array}$$

The operation requires a little explanation. Dividing 35 by 7, we get 5, and 6 over; for 35 in the duodenary scale is $3 \times 12 + 5$ or 41 in the common scale. ●

Then dividing $6e$ by 7 we get e , and 6 over, for

$$6e = 6 \times 12 + 11 = 83. \text{ And so on.}$$

In the next examples, the student is recommended to work out, as above, the process for himself.

Ex. 3. The numbers 357 and 76 are in the duodenary scale; find their product.

$$\begin{array}{r} 357 \\ 76 \\ \hline 1896 \\ 2031 \\ \hline 21et6 \end{array}$$

Ex. 4. Find the quotient of 1420422 divided by 241, both numbers being in the senary scale.

$$\begin{array}{r} 241)1420422(3502 \\ 1203 \\ \hline 2134 \\ 2125 \\ \hline 522 \\ 522 \\ \hline \end{array}$$

Ex. 5. Find the area of a room which measures 27 ft. 9 in. by 10 ft. 10 in.

Expressing these in the duodenary scale, we have

$$\begin{array}{r} 23.9 \\ 16.t \\ \hline 1el6 \\ 11t6 \\ 239 \end{array}$$

$$\bullet \quad 376.76 = 376.76 \text{ sq. ft. in the duodenary scale.}$$

This expressed in the common scale

$$= (3 \cdot 12^2 + 7 \cdot 12 + 6) \text{ sq. ft. } (7 \times 12 + 6) \text{ sq. in.} \\ = 522 \text{ sq. ft. } 90 \text{ sq. in.}$$

216. *To transpose a given proper fraction from any scale of notation to any other scale.*

If we transform the numerator and denominator each into the new scale, we transpose the fraction into the new scale.

It may, however, be required to transform it into a fraction having the form of a decimal in ordinary notation. Such fractions are called **radix fractions**.

We proceed to investigate the method of doing this.

Let F be the given proper fraction,

And let q_1, q_2, q_3 , &c., be the digits of the radix fraction, commencing from the left.

We then have

$$F = \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{q_3}{r^3} + \&c. \dots \dots \dots (1).$$

where q_1, q_2, q_3 , &c., are integers whose values are required.

Multiplying each side of (1) by r , we have

$$Fr = q_1 + \frac{q_2}{r} + \frac{q_3}{r^2} + \&c. \dots \dots \dots (2).$$

Hence, the first digit, q_1 , of the radix fraction is found by multiplying the given fraction by the proposed radix, and taking the integral portion.

The remaining fractional part is $\frac{q_2}{r} + \frac{q_3}{r^2} + \&c.$, and it is therefore evident that the second digit q_2 of the radix fraction is found by multiplying this by the radix. And so on.

Hence we have the following rule for bringing a given fraction to a radix fraction.

RULE.—*Multiply the given fraction by the radix, and the integral portion of the product is the first digit. Multiply the fractional part by the radix, and the integral portion of the product is the second digit. And so on.*

Ex. 1. Transform $\cdot 8125$ from the common scale to the scale whose radix is 8.

$$\begin{array}{r}
 \cdot 8125 \\
 \underline{8} \\
 6 \cdot 5000, \\
 \underline{8} \\
 4 \cdot 0
 \end{array}
 \quad
 \begin{array}{l}
 \therefore q_1 = 6. \\
 \\
 \therefore q_2 = 4.
 \end{array}$$

Hence the required radix fraction is $\cdot 64$.

Proof.— $\cdot 64$, in the scale whose radix is 8, $= \frac{6}{8} + \frac{4}{8^2}$
 $= \cdot 75 + \cdot 0625 = \cdot 8125$.

Ex. 2. Transform $713 \cdot 44$ in the scale eight to the scale six.

$$\begin{array}{r|l}
 6 & 713 \\
 \hline
 6 & 114 \dots 3 = p_0, \\
 \hline
 6 & 14 \dots 4 = p_1, \\
 \hline
 6 & 2 \dots 0 = p_2, \\
 \hline
 & \dots 2 = p_3.
 \end{array}
 \quad
 \begin{array}{r}
 \cdot 44 \\
 \underline{6} \\
 3 \cdot 30, \therefore q_1 = 3. \\
 \underline{6} \\
 2 \cdot 2, \therefore q_2 = 2. \\
 \underline{6} \\
 1 \cdot 4, \therefore q_3 = 1. \\
 \underline{6} \\
 3 \cdot 0, \therefore q_4 = 3.
 \end{array}$$

Hence the number required is $2043 \cdot 3213$.

217. A number whose radix is r is divisible by $r-1$, when the sum of its digits is so divisible.

Let the number N be such that

$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots + p_2r^2 + p_1r + p_0 \dots (1)$,
 where $p_0, p_1, p_2, \dots, p_{n-1}$ are the digits commencing at the right hand.

We have

$$N = p_{n-1}(r^{n-1} - 1) + p_{n-2}(r^{n-2} - 1) + \dots + p_2(r^2 - 1) \\ + p_1(r - 1) + (p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0).$$

Now each of the quantities $r^{n-1} - 1, r^{n-2} - 1, \dots, r^2 - 1, r - 1$, is divisible by $r - 1$. Hence the whole of the right side of this equation is divisible by $r - 1$, if $(p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0)$ is so divisible; i.e.,

N is divisible by $r - 1$, when $(p_{n-1} + p_{n-2} + \dots + p_2 + p_1 + p_0)$, the sum of its digits, is so divisible.

COR. Hence also, a number whose radix is r , will leave the same remainder when divided by $r - 1$, as the sum of its digits leaves when divided by $r - 1$.

218. *A number whose radix is r is divisible by $r + 1$, when the difference between the sum of the digits in the odd places and the sum of the digits in the even places is so divisible.*

We have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \dots + p_3r^3 + p_2r^2 + p_1r + p_0 \dots (1). \\ = p_{n-1}\{r^{n-1} - (-1)^{n-1}\} + p_{n-2}\{r^{n-2} - (-1)^{n-2}\} + \dots \\ + p_3(r^3 + 1) + p_2(r^2 - 1) + p_1(r + 1) \\ + \{p_0 - p_1 + p_2 - p_3 + \dots + (-1)^{n-1}p_{n-1}\}.$$

Now (Vol. I., Art. 30) each of the quantities $r + 1, r^2 - 1, r^3 + 1, \dots, r^{n-1} - (-1)^{n-1}$, is divisible by $r + 1$.

Hence the whole of the right side of this equation is divisible by $r + 1$, if $\{p_0 - p_1 + p_2 - p_3 + \dots + (-1)^{n-1}p_{n-1}\}$ is so divisible.

But $p_0 - p_1 + p_2 - p_3 + \dots + (-1)^{n-1}p_{n-1}$ is the difference between the sums of the digits in the odd and even places.

Hence N is divisible by $r + 1$, when the difference between the sums of the digits in the odd and even places is so divisible.

COR. Hence, also, when a number, whose radix is r , is divided by $r + 1$, it will leave the same remainder as the difference between the sums of the digits in the odd and even places will leave when so divided.

Ex. XLIII.

1. Express the common numbers 654 and 387 in the senary scale.

2. Transform 372 and 516 from the denary to the septenary scale, multiply the results together, and divide the product by 516 (septenary scale). Express the quotient in the denary scale.

3. Transform 4163 from the septenary to the scale whose radix is eight.

4. The number 86 expressed in another scale is 95. Find the radix.

5. The number 172 expressed in a different scale is 124. What is the radix.

6. If the number 516 be divided by 2, the quotient when expressed in another scale has the same digits as the original number. Find the scale.

7. Transform 79·875 from the denary scale to the scale whose radix is 8.

8. Express 4521 (senary scale) in the duodenary scale.

9. Find the area of a room measuring 52 ft. 9 in. by 30 ft. 7 in., expressing your result in the duodenary scale.

10. Simplify $\frac{313 \times 1034\frac{1}{2}}{45\frac{5}{16} \times 210}$, the number being in scale six. Give your answer in scale seven.

11. The number 52614 is in scale seven. Find its square root in that scale.

12. Reduce $\cdot\dot{3}\dot{2}$ (scale six) to a vulgar fraction in the same scale.

13. Show that 1331 is a cube number in any scale above three.

14. The difference between any number and the number composed of the same digits in reverse order is divisible by $r - 1$, where r is the radix of the scale.

15. Show that the difference between two numbers having the same digits is divisible by $r - 1$, whatever be the order of the digits in the second number.

16. Given a system of weights 1 lb., 3 lbs., 3² lbs., &c., (one of each kind); show how to weigh 1135 lbs.

17. If $\phi(n)$ represent the remainder after dividing a number n by $r - 1$, show that

$$\phi\{\phi(p) \cdot \phi(q)\} = \phi(pq).$$

18. If the digits of a number n be added together, the digits of the result added together, and so on, until the sum is represented by one digit; and we represent the final result by $F(n)$, show that

$$F(n) \cdot F(p) \cdot F(q) \dots \text{and } F(npq \dots)$$

are either equal or differ by a multiple of $(r - 1)$.

19. If a number whose digits are p_0, p_1, p_2 , is a perfect square in any scale, find the relation between p_0, p_1, p_2 .

20. Show that if the sum of two numbers is a multiple of the radix, the sum of the last digits of their cubes is equal to the radix.

21. Every number having its digits repeated in threes, and being in any scale of the series—*four, seven, ten, thirteen*, &c.—is divisible by 3, and the quotient contains a factor, the sum of whose digits is equal to the radix.

22. The sum of the numbers (scale s) represented by the permutations of any three digits of the scale amongst each other, is always divisible by the number ($s + 1$) of the scale whose radix is $(s + 1)$.

23. Every number less than 2^{n+1} can be expressed by the sum of certain terms of the series

$$1, 2, 2^2, \dots, 2^n.$$

24. Every number less than 3^{n+1} can be expressed by terms of the series

$$1, 3, 3^2, \dots, 3^n,$$

the coefficients of the terms being either ± 1 .

CHAPTER XXV.

Properties of Numbers.

219. By the term number in the present chapter we mean a positive integer.

Owing to the extensive nature of the Theory of Numbers we shall be able to give in this work only the more elementary propositions.

A Prime number is one which has no divisor except itself and unity.

Two numbers are said to be prime to each other when they contain no other common divisor than unity.

220. *If a number a be prime to each of the numbers b and c, it is prime to their product bc.*

For suppose a, b, c to be each broken up into their elementary factors.

Then a contains no factor except unity, which b contains; and neither does it contain any factor except unity, which c contains.

Now, bc when broken up into elementary prime factors cannot contain any other such factors than what can be found in b and c .

Therefore a contains no factor common to bc other than unity.

Hence a is prime to bc .

COR. 1. Hence if a be prime to any number of quantities $b, c, d, \&c.$, it is prime to their product $bcd, \&c.$; and if a be prime to any quantity b , it is prime to each of the factors of b .

COR. 2. If a be prime to b , it is also prime to $bbb \dots$, that is, to any power of b .

COR. 3. If a be a divisor of bc , and be prime to b , it must be a divisor of c .

COR. 4. If $\frac{a}{b}$ is a fraction in its lowest terms, then $\frac{a^n}{b^n}$ is in its lowest terms.

221. If $\frac{a}{b}$ be a fraction in its lowest terms, and $\frac{a}{b} = \frac{c}{d}$; then will c and d be equimultiples of a and b .

For $\frac{a}{b}$ is the fraction $\frac{c}{d}$ reduced to its lowest terms.

Now in order to express $\frac{c}{d}$ in its lowest terms we have to divide numerator and denominator by the greatest common measure x , suppose.

We must therefore have $\frac{c}{x} = a$, or $c = ax$;

and $\frac{d}{x} = b$, or $d = bx$.

222. If N be an integer, and $\sqrt[n]{N}$ a surd, we cannot express the latter by a fraction.

For suppose $\sqrt[n]{N} = \frac{a}{b}$, we then have $N = \frac{a^n}{b^n}$, ... (1).

Now if $\frac{a}{b}$ be not in its lowest terms, we may so reduce it.

Then we shall have $\frac{a^n}{b^n}$ in its lowest terms, and also a fraction.

Hence in (1) we have, an integer = a fraction, which is absurd.

223. Any number is prime if it cannot be divided by some number equal to, or less than, its square root.

For every number N which is not a prime is composed of two factors, as a and b , so that we have

$$N = ab.$$

Now if $a = b$, we have $a = b = \sqrt{N}$, so that N is divisible by $\sqrt{N}$.

Again, if $a > b$, we have $b < \sqrt{N}$;

And, if $a < b$, we have $a < \sqrt{N}$.

Hence, for every number not prime, there always exists a divisor, either equal to, or less than, its square root; and therefore, every number not having such a divisor is prime.

Ex. To determine whether 101 is a prime number.

We have $\sqrt{101} < 11$.

Now, as 101 is odd, we need only try the prime divisors 3, 5, 7; and we find it is not divisible by either of them.

Hence 101 is a prime number.

224. *No rational algebraical formula can contain prime numbers only.*

For, if possible, suppose that the expression

$$p + qx + rx^2 + sx^3 + \&c. \dots$$

expresses prime numbers only.

Suppose that when $x = m$, the expression equals P ; and when $x = m + nP$, the value of the expression is P_1 .

Then we have

$$\begin{aligned} P &= p + qm + rm^2 + sm^3 + \&c. \dots \dots \dots (1), \\ \text{and } P_1 &= p + q(m + nP) + r(m + nP)^2 + s(m + nP)^3 + \&c. \\ &= (p + qm + rm^2 + sm^3 + \&c.) \\ &\quad + (qn + 2rmn + 3sm^2n + \&c.)P \\ &\quad + (rn^2 + 3smn^2 + \&c.)P^2 + \&c. \\ &= P + \text{terms containing } P \text{ as a factor.} \end{aligned}$$

Hence P_1 is divisible by P , and therefore not prime.

And hence the algebraical expression

$$p + qx + rx^2 + sx^3 + \&c.$$

cannot represent prime numbers only.

REMARK. There are several remarkable algebraical formulæ which represent a large number of primes.

Thus $x^2 + x + 41$, by making $x = 0, 1, 2, 3, \&c. 39$, will give a series of forty numbers, all primes.

$x^2 + x + 17$ gives similarly seventeen primes.

$2x^2 + 29$ gives twenty-nine primes.

$2^n + 1$ is prime for $n = 1, 2, 4, 8$, or 16.

225. *The number of primes is infinite.*

For suppose p to be the greatest prime number. Then the product of all the prime numbers up to p is

$$2 \cdot 3 \cdot 5 \cdot 7 \dots p.$$

And this product is divisible by each of the prime numbers. Hence

$$(2 \cdot 3 \cdot 5 \cdot 7 \dots p) + 1$$

is divisible by none of the prime numbers $2, 3, 5, 7, \dots p$.

Now, if this quantity is a prime, it is a greater prime than p ; and if it be a composite number, it must have a prime factor greater than p , since none of the prime numbers $2, 3, 5, 7, \dots p$ will divide it.

Hence, in any case, there exists a prime number greater than any supposed prime p .

226. *If a be prime to b , and each of the terms of the series*

$$b, 2b, 3b, \dots (a-1)b$$

be divided by a , all the remainders will be different.

For suppose any two terms of this series, mb and m_1b , to have the same remainder, where m and m_1 are of course each less than a .

Then their difference $(mb - m_1b)$ will be divisible by a .

But since $mb - m_1b = (m - m_1)b$, it is composed of two factors, one of which is prime to a , and the other of which is less than a . It therefore cannot be divisible by a .

Hence no terms of the given series can leave the same remainder when divided by a .

COR. 1. Since all these remainders are less than a , and they are all different, and since there are $(a - 1)$ terms in the given series, it follows that the series

$$1, 2, 3, \dots, a - 1$$

represents the remainders, not regarding their order.

COR. 2. If a be prime to b , the terms of the A. P.

$$b + c, 2b + c, 3b + c, \dots, (a - 1)b + c,$$

when divided by a , will give for remainders the series

$$1, 2, 3, \dots, (a - 1),$$

no regard being had to their order.

Fermat's Theorem.

227. If n be a prime number, and N be a number prime to n , then $N^{n-1} - 1$ is divisible by n .

We shall give two demonstrations of this important theorem.

First proof.

By Art. 226, Cor. 1, the terms of the series

$$N, 2N, 3N, \dots, (n - 1)N,$$

when divided by n , give for remainders (disregarding their order) the series

$$1, 2, 3, \dots, (n - 1).$$

Hence, assuming $q_1, q_2, q_3, \dots, q_{n-1}$ as the quotients corresponding to these remainders, we must have

$$N \cdot 2N \cdot 3N \dots (n - 1)N$$

$$= (1 + q_1n) (2 + q_2n) (3 + q_3n) \dots (\overline{n - 1} + q_{n-1}n); \text{ or,}$$

$$N^{n-1} \cdot 1 \cdot 2 \cdot 3 \dots (n - 1) = 1 \cdot 2 \cdot 3 \dots (n - 1) + \text{terms containing } n \text{ as a factor}$$

$$= 1 \cdot 2 \cdot 3 \dots (n - 1) + Mn, \text{ suppose.}$$

$$\therefore (N^{n-1} - 1) \cdot 1 \cdot 2 \cdot 3 \dots (n - 1) = Mn.$$

Now n , being a prime number, is prime to each of the factors $1, 2, 3, \dots, (n-1)$. And hence

$\frac{M}{1 \cdot 2 \cdot 3 \dots (n-1)}$ is an integer,
since $N^{n-1} - 1$ and n are integers.

Hence we have $N^{n-1} - 1 = Pn$, when P is integral.

$\therefore N^{n-1} - 1$ is divisible by n .

Second proof.

We have $N^n = \{(N-1) + 1\}^n =$
 $(N-1)^n + n(N-1)^{n-1} + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r} (N-1)^{n-r}$
 $+ \dots + n(N-1) + 1.$

Now every term of the second side of this identity, excepting the first and last, is divisible by n .

For the general coefficient $\frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}$ is an integer, and is so divisible, since r is $< n$, and n is prime.

Hence we have

$N^n = (N-1)^n + 1 + P_0 n$, where P_0 is an integer.

$\therefore N^n - N = (N-1)^n - (N-1) + P_0 n$;

or, putting successively, $N-1, N-2, \dots, 2$, for N , we have

$(N-1)^n - (N-1) = (N-2)^n - (N-2) + P_1 n$,

$(N-2)^n - (N-2) = (N-3)^n - (N-3) + P_2 n$,

&c. = &c.

$2^n - 2 = 1^n - 1 + P_{n-2} n$

where P_1, P_2 , &c., are integers.

Hence adding these equations, we have—

$N^n - N = 1^n - 1 + (P_0 + P_1 + P_2 + \dots + P_{n-2})n$
 $= Pn$, suppose, where P is an integer.

Or, $N(N^{n-1} - 1) = Pn$.

Now n is prime to N , and therefore $\frac{P}{N}$ is integral.

Hence $N^{n-1} - 1$ is divisible by n .

Extension of Fermat's Theorem.

228. If n be prime to N , and $\phi(n)$ represent how many numbers there are less than n , and prime to it, then

$N^{\phi(n)} - 1$ is divisible by n .

Let $a, b, c, d, \dots, m, \dots$(1).

be the $\phi(n)$ numbers which are less than n and prime to it. Then if the series be multiplied by N we have the second series

$aN, bN, cN, \dots, mN, \dots$(2).

Now it may be shown, as in Art. 226, that if the terms of the series be severally divided by n , we shall have for remainders, neglecting their order, the series (1).

Hence, as in Art. 227, assuming q_a, q_b, q_c &c., q_n as the quotients corresponding to these remainders, we must have

$$aN \cdot bN \cdot cN \dots mN$$

$$= (a + q_1n) (b + q_2n) (c + q_3n) \dots (m + q_mn); \text{ or,}$$

$$abc \dots m \cdot N^{\phi(n)}$$

$$= abc \dots m + \text{terms containing } n \text{ as a factor}$$

$$= abc \dots m + Mn, \text{ suppose.}$$

$$\therefore \{N^{\phi(n)} - 1\} abc \dots m = Mn.$$

Now each of the quantities $a, b, c, \dots, m$ is prime to n and less than n . Hence n is prime to the product $abc \dots m$.

Therefore $N^{\phi(n)} - 1$ is divisible by n .

COR. Fermat's theorem follows at once from this.

For suppose n a prime number, then the numbers less than n and prime to it are

$$1, 2, 3, \dots, (n-1).$$

Hence, here $\phi(n) = n-1$.

Hence, if n be a prime number, and n be prime to N ,

$$N^{n-1} - 1 \text{ is divisible by } n.$$

Wilson's Theorem.

229. If n be a prime number, then

$$(n-1)! + 1$$

is divisible by n .

Let a represent one of the numbers of the series

$$1, 2, 3, \dots, (n-1) \dots \dots \dots (1).$$

a is therefore prime to n .

Multiplying each term of this series by a , we have the second series

$$a, 2a, 3a, \dots, (n-1)a \dots \dots \dots (2).$$

Now (Art. 226, Cor. 1) if the terms of this series be divided by n , we have for remainders, not regarding their order, the series in (1).

Hence some one of the numbers in series (2) give *unity* for a remainder when divided by n .

If we represent this number by ma , we have

$$ma - 1 \text{ divisible by } n = b_0n, \text{ suppose } \dots \dots \dots (3).$$

Now m cannot be equal to a , except

$$a = 1, \text{ or } a = n - 1 \dots \dots \dots (4).$$

For suppose we have $m = a$, we then have from (3) $a^2 - 1 = b_0n$ or $(a+1)(a-1) = b_0n$.

Now n is a prime number ;

Hence either $a+1$ or $a-1$ is divisible by n .

But a is less than n , and hence when n divides $a+1$, we must have $a = n-1$.

And so, when n divides $a-1$, we must have $a = 1$.

Hence, leaving out the numbers 1 and $n-1$, we may say that it is always possible that *any one* of the numbers

$$2, 3, 4, \dots, (n-2),$$

may be multiplied by *some other* number of the same series, so as to form a product of the form $b_0n + 1$.

Suppose then we multiply these numbers so as to form such products as $b_1n + 1$, $b_2n + 1$, &c. By multiplying together these products we have

$$2 \cdot 3 \cdot 4 \dots (n-2) = (b_1n + 1)(b_2n + 1) \dots \\ = Mn + 1,$$

where Mn represents the terms having n as a factor.

Hence, multiplying each side by $n-1$, we have

$$1 \cdot 2 \cdot 3 \dots (n-1) = (Mn + 1)(n-1) \\ = M(n-1)n + n-1; \text{ or}$$

$$1 \cdot 2 \cdot 3 \dots (n-1) + 1 = \{M(n-1) + 1\}n.$$

Hence $\lfloor \frac{n-1}{n} \rfloor + 1$ is divisible by n .

COR. Hence $\left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right)^2 \pm 1$ is divisible by n , when n is prime, and > 2 , the upper or lower sign being taken as n is of the form $4m+1$ or $4m-1$.

Since $\frac{n-1}{2}$ is integral, we have

$$1 \cdot 2 \cdot 3 \dots (n-1) \\ = \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right) \cdot \left(\frac{n+1}{2} \cdot \frac{n+3}{2} \dots \overline{n-3} \cdot \overline{n-2} \cdot \overline{n-1}\right) \\ = 1 \cdot 2 \cdot 3 \dots \frac{n-1}{2} \cdot \left\{\left(n - \frac{n-1}{2}\right) \dots \overline{n-3} \cdot \overline{n-2} \cdot \overline{n-1}\right\} \\ = 1 \cdot 2 \cdot 3 \dots \frac{n-1}{2} \cdot \left\{Mn + (-1)^{\frac{n-1}{2}} 1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right\}$$

where Mn represents terms containing n as a factor;

$$= \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right) Mn + (-1)^{\frac{n-1}{2}} \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right)^2;$$

or, adding unity to each side, $\lfloor \frac{n-1}{n} \rfloor + 1$

$$= \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right) Mn + (-1)^{\frac{n-1}{2}} \cdot \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right)^2 + 1.$$

Now we have just shown that $\lfloor \frac{n-1}{n} \rfloor + 1$ is divisible by n .

Hence $(-1)^{\frac{n-1}{2}} \left(1 \cdot 2 \cdot 3 \dots \frac{n-1}{2}\right)^2 + 1$ is divisible by n .

Now $(-1)^{\frac{n-1}{2}} = \pm 1$, according as n is of the form of $4n+1$ or $4n-1$.

Hence the truth of the proposition.

230. If $a, b, c, \dots, 1$ are a series of numbers prime to each other, and $\phi(n)$ represents how many numbers there are which are prime to n and less than it, then

$$\phi(abc \dots 1) = \phi(a) \cdot \phi(b) \cdot \phi(c) \dots \phi(1).$$

First, suppose $N = ab$.

We may arrange the numbers from 1 to ab in b columns thus :

$$\begin{array}{cccc} 1, & 2, & 3, \dots & k, \dots a, \\ a+1, & a+2, & a+3, \dots & a+k, \dots 2a, \\ 2a+1, & 2a+2, & 2a+3, \dots & 2a+k, \dots 3a, \\ \vdots & \vdots & \vdots & \vdots \\ (b-1)a+1, & (b-1)a+2, & (b-1)a+3, \dots & (b-1)a+k, \dots ab. \end{array}$$

If we consider any column as the one commencing with k , we see that all the numbers in it are prime to a , or all the numbers contain some factor common with a , according as k is prime to a or not.

Hence the number of columns which contain all the numbers of the column prime to a is equal to the number of quantities in the series $1, 2, 3, \dots k, \dots a$, which are prime to a . And no other column contains any number prime to a .

Hence $\phi(a)$ is the number of columns containing numbers prime to a out of all the numbers from 1 to ab(1).

Again, by Art. 226, Cor. 1, if the numbers in any column, as that commencing with k , be divided by b , the remainders, when k is prime to b , will form the series

$$1, 2, 3, \dots, (b-1).$$

Hence in every column there are as many numbers prime to b as there are of such numbers in the series $1, 2, 3, \dots (b-1)$.

Therefore $\phi(b)$ is the number of quantities in each column which are prime to b(2).

Hence from (1) and (2) we conclude that—

The number of quantities in all the columns which are prime to a and prime to b , and therefore prime to ab , is $\phi(a) \cdot \phi(b)$.

Therefore $\phi(ab) = \phi(a) \cdot \phi(b)$.

If $N = abc \dots l$, we have

$$\begin{aligned}\phi(abc \dots l) &= \phi(a) \cdot \phi(bc \dots l) \\ &= \phi(a) \cdot \phi(b) \cdot \phi(c \dots l) = \&c. \\ &= \phi(a) \cdot \phi(b) \cdot \phi(c) \dots \phi(l). \quad Q.E.D.\end{aligned}$$

NOTE. It will be seen that we consider here *unity* as prime to $a, b, c, \&c$.

231. To find the number of positive integers which are less than N and prime to it.

Suppose $N = a^p b^q c^r \dots l^s$, where $a, b, c, \dots, l$ are prime factors.

Now the number of positive integers which are less than a^p and prime to it is found by subtracting from a^p the number of quantities in the series

$$a, 2a, 3a, \dots, a^{p-1}a \text{ or } a^p.$$

The number of these is evidently a^{p-1} .

Hence we must have $\phi(a^p) = a^p - a^{p-1}$,

$$\text{or } \phi(a^p) = a^p \left(1 - \frac{1}{a}\right); \text{ and so}$$

$$\phi(b^q) = b^q \left(1 - \frac{1}{b}\right),$$

$$\phi(c^r) = c^r \left(1 - \frac{1}{c}\right),$$

$$\&c., = \&c.$$

$$\phi(l^s) = l^s \left(1 - \frac{1}{l}\right).$$

Hence, multiplying, we have

$$\begin{aligned} & \phi(a^p) \cdot \phi(b^q) \cdot \phi(c^r) \dots \phi(l^s) \\ &= a^p b^q c^r \dots l^s \cdot \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots \left(1 - \frac{1}{l}\right) \\ &= N \cdot \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots \left(1 - \frac{1}{l}\right). \end{aligned}$$

But by last Art.,

$$\begin{aligned} \phi(N) &= \phi(a^p) \cdot \phi(b^q) \cdot \phi(c^r) \dots \phi(l^s). \\ \text{Hence } \phi(N) &= N \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots \left(1 - \frac{1}{l}\right). \end{aligned}$$

232. To find the number of divisors in a given integer.

Suppose $N = a^p b^q c^r \dots$, where $a, b, c \dots$ are primes.

Now N is divisible by every power of a not greater than a^p .

Hence, including *unity* as one of its divisors, N is divisible by every term of the series

$$1, a, a^2, \dots, a^p.$$

And so it is divisible by every term of each of the series

$$1, b, b^2, \dots, b^q,$$

$$1, c, c^2, \dots, c^r,$$

&c.

N is therefore divisible by every combination of the terms of these series.

Hence N is divisible by every term of the product

$$\begin{aligned} & (1 + a + a^2 + \dots + a^p) (1 + b + b^2 + \dots + b^q) \\ & (1 + c + c^2 + \dots + c^r) \dots, \end{aligned}$$

and by no other quantity.

Now the *number of terms* of this product is found by putting $a = b = c = \&c. = 1$.

Therefore, number of divisors of N , including itself and unity

$$= (p + 1) (q + 1) (r + 1) \dots$$

COR. 1. Since every divisor has a corresponding divisor for a quotient, it follows that the number of ways in which N may be broken into two factors is half the number of divisors.

(i) When N is *not* a perfect square,

One at least of the indices is odd, and therefore

$$(p + 1) (q + 1) (r + 1) \dots$$

is even.

Hence, number of ways in which N may be broken up into two factors .

$$= \frac{1}{2} (p + 1) (q + 1) (r + 1) \dots$$

(ii.) When N is a perfect square,

All the indices, $p, q, r, \dots$ are even, and therefore

$$(p + 1) (q + 1) (r + 1) \dots$$

is odd.

But $\sqrt{N}$ is a divisor, and hence, since $N = \sqrt{N} \cdot \sqrt{N}$, there is *one way* of resolving N into two factors, by means of *one divisor*.

Hence, number of ways in which N may be resolved into two factors

$$= \frac{1}{2} \{ (p + 1) (q + 1) (r + 1) \dots + 1 \}.$$

It must be remembered that $(N, 1)$ is one of the pairs of factors.

COR. 2. The sum of the divisors of any number N , including among them itself and unity,

$$\begin{aligned} &= \text{the product } (1 + a + a^2 + \dots + a^p) (1 + b + b^2 + \dots + b^q) \\ &\quad (1 + c + c^2 + \dots + c^r) \dots \\ &= \frac{a^{p+1} - 1}{a - 1} \cdot \frac{b^{q+1} - 1}{b - 1} \cdot \frac{c^{r+1} - 1}{c - 1} \dots \end{aligned}$$

COR. 3. If n be the number of prime factors, a, b, c , &c. (not including N or unity), in N ,

The number of ways in which N may be resolved into two factors prime to each other $= 2^{n-1}$.

For we must have a^p in one factor of any pair; and so we must have $b^q, c^r, \&c.$; and we cannot have any lower powers of $a, b, c, \&c.$, in any of the factors which are prime to each other.

Hence the *number of ways* in which we may resolve N into two factors prime to each other is the same as that of the number $abc \dots$;

We have therefore, by Cor. 1, making there $p = q = r = \&c. = 1$,

Number of ways in which N may be resolved into two factors prime to each other,

$$= \frac{1}{2} (1 + 1) (1 + 1) (1 + 1) \dots \text{to } n \text{ factors} = 2^{n-1}.$$

EX. XLIV.

1. If any number divide each of two other numbers, it will divide their sum and difference.

2. If two numbers be prime to each other, then will their sum and difference be prime to each of them.

3. If a square number be multiplied by a square, the product will be a square.

4. All numbers belong to one of the forms $3n, 3n \pm 1$.

5. Every square number is of the form $5n$ or $5n \pm 1$.

6. Every cube number is of the form $7n$ or $7n \pm 1$.

7. A number which is a perfect square will divide by 5 without a remainder, or will leave 1 or 4 for a remainder.

8. Show that $n(n^2 + 5)$ is divisible by 6, when n is an integer.

9. Show that $n(n^2 - 1)(n^2 - 4)$ is divisible by 120, when n is an integer.

10. All even square numbers are of the form $16n$ or $4(4n + 1)$, and all odd squares of the form $8n + 1$.

11. Every number of the form $4n + 1$ is the sum of two squares.

12. If the difference between a and b is divisible by p , a is said to be congruous to b , for the modulus p . Hence show that if $a \equiv b \pmod{p}$, and $c \equiv d \pmod{p}$, then $a + c \equiv b + d \pmod{p}$.

13. If $a \equiv b \pmod{p}$, then $a^n \equiv b^n \pmod{p}$, and $ma \equiv mb \pmod{p}$.

14. If $ma \equiv mb \pmod{p}$, what is the condition that $a \equiv b \pmod{p}$?

15. If the congruence $\phi(x) \equiv 0 \pmod{p}$ is satisfied by $x = a$, show that every number of the form $a + mp$ is also a solution, and that among these numbers there always exists one, and only one, between the limits 0 and $p - 1$ inclusive.

16. If $ax + b \equiv 0 \pmod{p}$, how many solutions are possible in the three cases: (i.) when a is prime to p ; (ii.) when a and p have a common measure, which will not divide b ; (iii.) when a, b, p have a common measure?

17. If p be a prime number, show that

$$(x + 1)(x + 2)(x + 3) \dots (x + p - 1) \\ \equiv x^{p-1} - 1 \pmod{p},$$

and hence deduce Wilson's theorem.

18. If $2^n - 1$ be a prime number, then will $2^{n-1}(2^n - 1)$ be a perfect number—that is, a number which is equal to the sum of all its aliquot parts, or its divisors.

19. If $3 \cdot 2^n - 1, 6 \cdot 2^n - 1, 18 \cdot 2^{2n} - 1$ are prime numbers, show that

$2^{n+1}(18 \cdot 2^{2n} - 1)$, and $2^{n+1}(3 \cdot 2^n - 1)(6 \cdot 2^n - 1)$, are amicable numbers—that is, numbers each of which is equal to the sum of the divisors of the other.

20. Any power of the sum of two squares may be separated into the sum of two squares.

21. If a, b , and n are positive integers, and b less than $2a - 1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

22. If n be a prime number, and m be odd, show that $m^n + 4$ cannot be a perfect square.

23. If $\phi(N)$ is the number of integers not greater than N and prime to it, and $1, d_1, d_2, \&c.$, are all the divisors of N , then

$$\phi(1) + \phi(d_1) + \phi(d_2) + \dots = N.$$

24. If p be the G. C. M. of m and n , then $x^p - 1$ is the G. C. M. of $x^m - 1$ and $x^n - 1$.

25. If p be a prime number, and a one of the numbers $1, 2, 3, \dots, p-1$, then $a + kp$ contains square numbers or not according as $a^{\frac{p-1}{2}} - 1$, or $a^{\frac{p-1}{2}} + 1$, is divisible by p .

26. With the notation of Ex. 23, show that, when N is an odd number, $\phi(2N) = \phi(N)$.

27. Find the sum of the positive integers which are less than a given number (N) and prime to it.

28. Show that the sum of the squares of these integers

$$= \frac{N^3}{3} \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots + \frac{N}{6} (1-a)(1-b)(1-c) \dots;$$

and that the sum of their cubes

$$= \frac{N^3}{4} \left\{ N^2 \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \left(1 - \frac{1}{c}\right) \dots + (1-a)(1-b)(1-c) \dots \right\},$$

where $a, b, c, \dots$, are the prime factors of N .

29. If p be a prime number, then the congruence

$$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_m \equiv 0 \pmod{p}$$

cannot admit of more than m solutions less than p .

30. The congruence $x^p \equiv 1 \pmod{p}$ has always $\phi(p-1)$ roots, where $\phi(n)$ represents the number of integers less than n and prime to it, and where p is a prime number.

CHAPTER XXVI.

DETERMINANTS.

233. We close with a short chapter as an introduction to Determinants. For further information the student is referred to special treatises on the subject.

234. When we eliminate x and y from the equations

$$a_1x + b_1y = 0,$$

$$a_2x + b_2y = 0,$$

we obtain the relation $a_1b_2 - a_2b_1 = 0$.

The expression $a_1b_2 - a_2b_1$ is called the **eliminant** of the given equations; and the condition that the given equations may have simultaneous values of x and y is that the eliminant must vanish.

It is convenient to express this relation thus—

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The expression on the first side of this equation is called a **determinant**.

We have then generally, according to this notation,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

We have also, on the same principle,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

therefore

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence the value of the determinant is not altered by making the rows columns, and columns rows.

This we shall see is a general property, whatever be the number of rows and columns.

The quantities a_1, b_1 , &c., are called the **constituents**, and the products a_1b_2 , &c., the **elements** of the determinant.

235. Suppose we have the two equations

$$a_1x + a_2y + a_3z = 0 \dots\dots\dots(1),$$

$$b_1x + b_2y + b_3z = 0 \dots\dots\dots(2).$$

We may express the mutual ratios of x, y, z in the notation of determinants thus—

Multiplying each side of (1) by b_2 , and of (2) by a_2 , we at once

$$(a_1b_2 - a_2b_1)x = (a_2b_3 - a_3b_2)z,$$

or

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} z.$$

therefore,

$$\frac{x}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

And so we therefore get

$$\frac{x}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \dots\dots\dots(3).$$

COR. 1. If we put each of these ratios = m , and substitute the corresponding value of x, y, z in the given equations, we obtain the identities—

$$a_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

$$\text{and} \quad b_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

X

COR. 2. If we put $z = 1$ in the given equations, they become

$$a_1x + a_2y + a_3 = 0, \text{ and } b_1x + b_2y + b_3 = 0,$$

and we get from (3) the values of x and y satisfying these equations, viz. :—

$$x = \frac{a_2b_3 - a_3b_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_3b_1 - a_1b_3}{a_1b_2 - a_2b_1}.$$

236. Taking the system of three equations

$$a_1x + b_1y + c_1z = 0 \dots\dots\dots(1),$$

$$a_2x + b_2y + c_2z = 0 \dots\dots\dots(2),$$

$$a_3x + b_3y + c_3z = 0 \dots\dots\dots(3),$$

And multiplying (1) by

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad (2) \text{ by } \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \quad (3) \text{ by } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

we get, remembering the identities proved in Cor. 1 of the last Art.,

$$c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

This relation may also be written

$$b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} - b_2 \begin{vmatrix} c_1 & a_1 \\ c_3 & a_3 \end{vmatrix} + b_3 \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = 0,$$

$$\text{or} \quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0.$$

The *expression* on the first side of these equations is expressed in the language of *determinants* by the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

When the value of this determinant vanishes we have just seen it to be the condition that x, y, z satisfy the simultaneous equations (1), (2), (3).

Cor. Since we have *identically*—

$$\begin{aligned} & a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2), \end{aligned}$$

it follows at once that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

237. Using the same notation, we have

$$\begin{aligned} & \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} \\ &\quad - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}. \end{aligned}$$

PROPERTIES OF DETERMINANTS.

238. If any two rows (or columns) be interchanged, the value of the determinant is changed in SIGN. We have—

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} &= a_1b_2 - a_2b_1. \\ &= -(a_2b_1 - a_1b_2) = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}. \end{aligned}$$

Again,

$$\begin{aligned}
 \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2, & c_2 \\ a_3, & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} \\
 &= -b \begin{vmatrix} a_2, & c_2 \\ a_3, & c_3 \end{vmatrix} + a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - c_1 \begin{vmatrix} b_2, & a_2 \\ b_3, & a_3 \end{vmatrix} \\
 &= - \left\{ b_1 \begin{vmatrix} a_2, & c_2 \\ a_3, & c_3 \end{vmatrix} - a_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2, & a_2 \\ b_3, & a_3 \end{vmatrix} \right\} \\
 &= - \begin{vmatrix} b_1, & a_1, & c_1 \\ b_2, & a_2, & c_2 \\ b_3, & a_3, & c_3 \end{vmatrix}.
 \end{aligned}$$

And we may extend the principle to a determinant of any order, the rule being that every single permutation of two rows or columns requires a change of sign.

239. *If two rows (or columns) of a determinant are identical the determinant vanishes.*

For, by the last article, if we interchange these *identical rows or columns*, the sign of the determinant is changed. But there is actually no change in the determinant by interchanging identical columns or rows. Hence the determinant has the same value whether positive or negative, and must therefore be zero.

240. *If the constituents of any row (or column) of a determinant be multiplied by the same constant factor, the determinant is thereby multiplied by the same factor.*

Thus

$$\begin{aligned}
 \begin{vmatrix} ma_1, & mb_1, & mc_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} &= ma_1 \begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - mb_1 \begin{vmatrix} a_2, & c_2 \\ a_3, & c_3 \end{vmatrix} \\
 &+ mc_1 \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} = m \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.
 \end{aligned}$$

COR. 1.—Hence, *when the constituents of any row or column contain a constant factor, that factor may be placed as a coefficient to the determinant formed by striking the common factor out of the row or column.*

COR. 2.—Since

$$\begin{vmatrix} ma_1 & a_1 & b_1 \\ ma_2 & a_2 & b_2 \\ ma_3 & a_3 & b_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0, \text{ it follows that,}$$

when any row (or column) of a determinant may be obtained from any other row (or column) by a CONSTANT MULTIPLIER, the determinant vanishes.

241. *If to the constituents of any row (or column) there be added the corresponding constituents of any other row (or column), multiplied by a constant factor, the value of the determinant is not altered.*

Thus

$$\begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ = (a_1 + ma_2) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (b_1 + mb_2) \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (c_1 + mc_2) \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

But, Article 230,

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Hence

$$\begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We shall now work out a few examples to show how to evaluate determinants, and how they may be practically applied.

Ex. 1.—Find the value of the determinant—

$$\begin{vmatrix} 6, & 4 \\ 2, & -1 \end{vmatrix}.$$

$$\text{Value required} = 6(-1) - 2(4) = -14.$$

Ex. 2.—Find the value of

$$\begin{vmatrix} 1, & 2, & 3 \\ 3, & -1, & 4 \\ 4, & 2, & 5 \end{vmatrix}.$$

$$\begin{aligned} \text{Value} &= 1 \begin{vmatrix} -1, & 4 \\ 2, & 5 \end{vmatrix} - 2 \begin{vmatrix} 3, & 4 \\ 4, & 5 \end{vmatrix} + 3 \begin{vmatrix} 3, & -1 \\ 4, & 2 \end{vmatrix} \\ &= 1(-13) - 2(-1) + 3(6+4) = 19. \end{aligned}$$

Ex. 3.—Find the value of

$$\begin{vmatrix} 48, & 6, & 12, & 18 \\ 30, & 6, & 6, & 12 \\ 4, & 4, & 6, & 10 \\ 20, & 20, & 5, & 25 \end{vmatrix}.$$

$$\begin{aligned} \text{Value} &= 6 \times 3 \times 2 \times 5 \cdot \begin{vmatrix} 8, & 1, & 2, & 3 \\ 10, & 2, & 2, & 4 \\ 2, & 2, & 3, & 5 \\ 4, & 4, & 1, & 5 \end{vmatrix} \\ &= 6 \times 3 \times 2 \times 5 \times 2 \cdot \begin{vmatrix} 4, & 1, & 2, & 3 \\ 5, & 2, & 2, & 4 \\ 1, & 2, & 3, & 5 \\ 2, & 4, & 1, & 5 \end{vmatrix}; \end{aligned}$$

add the 2nd and 3rd columns together for a new 3rd column :

$$= 360 \cdot \begin{vmatrix} 4, & 1, & 3, & 3 \\ 5, & 2, & 4, & 4 \\ 1, & 2, & 5, & 5 \\ 2, & 4, & 5, & 5 \end{vmatrix} = 0, \text{ by Article.}$$

Ex. 4.—Solve the simultaneous equations—

$$3x - y - 2z = -1.$$

$$2x - 4y + 5z = 9.$$

$$x + 3y - 6z = -11.$$

If we suppose $u = 1$, we have

$$3x + y - 2z + u = 0.$$

$$2x - 4y + 5z - 9u = 0.$$

$$x + 3y - 6z + 11u = 0.$$

Hence, as in Article 235,

$$\begin{aligned} \frac{x}{\begin{vmatrix} 1, & -2, & 1 \\ -4, & 5, & -9 \\ 3, & -6, & 11 \end{vmatrix}} &= \frac{y}{\begin{vmatrix} 3, & -2, & 1 \\ 2, & 5, & -9 \\ 1, & -6, & 11 \end{vmatrix}} \\ &= \frac{z}{\begin{vmatrix} 1, & 3, & 1 \\ -9, & 2, & -4 \\ 11, & 1, & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3, & 1, & -2 \\ 2, & -4, & 5 \\ 1, & 3, & -6 \end{vmatrix}}. \end{aligned}$$

The value of the determinants are respectively

$$-24, -48, -72, -24.$$

Therefore

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{-72} = \frac{1}{-24}.$$

therefore

$$x = 1, y = 2, z = 3.$$

Ex. 5.—If

$$\frac{a_1 - b_1}{b_1 - c_1} = \frac{a_2 - b_2}{b_2 - c_2} = \frac{a_3 - b_3}{b_3 - c_3}, \text{ then } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Let m be the value of each of the given ratios, then

$$a_1 - b_1 = m(b_1 - c_1) \dots \dots \dots (1).$$

$$a_2 - b_2 = m(b_2 - c_2) \dots \dots \dots (2).$$

$$a_3 - b_3 = m(b_3 - c_3) \dots \dots \dots (3).$$

Now, Article 232,

$$\begin{aligned} & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} m(b_1 - c_1) & m(b_2 - c_2) & m(b_3 - c_3) \\ b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= m \begin{vmatrix} b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = m(0) = 0. \end{aligned}$$

Ex. XLV.

1. Find the value of the determinants—

$$\begin{vmatrix} 3 & 0 & 1 \\ 4 & 2 & 7 \\ 6 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 6 & -1 & 5 & 5 \\ 1 & 1 & 0 & 2 \\ -1 & 4 & 8 & 3 \\ 7 & 5 & 7 & 4 \end{vmatrix}, \begin{vmatrix} 4 & 0 & 1 & -2 \\ 6 & 4 & 1 & 0 \\ -1 & 2 & 4 & 1 \\ 6 & -1 & 7 & 0 \end{vmatrix}.$$

2. Evaluate the determinants

$$\begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix}, \text{ and } \begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix},$$

and show that their ratio is $\frac{abcd}{a+b+c+d}$.

3. If

$$x_2 + x_3 + x_4 + \dots + x_n = a_1,$$

$$x_1 + x_3 + x_4 + \dots + x_n = a_2,$$

$$\dots = \dots$$

$$x_1 + x_2 + x_3 + \dots + x_{n-1} = a_n,$$

show that

$$x_1 = \frac{1}{n-1} \{a_2 + a_3 + \dots + a_n - (n-2)a_1\},$$

$$x_2 = \frac{1}{n-1} \{a_1 + a_3 + \dots + a_n - (n-2)a_2\},$$

$$\dots = \dots$$

$$x_n = \frac{1}{n-1} \{a_1 + a_2 + \dots + a_{n-1} - (n-2)a_n\}.$$

4. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3.$$

5. If a, b, c be the sides of a triangle, show that the square of the area of the triangle

$$= -16 \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2a & -2b & -2c \\ 2a & 0 & 2c & 2b \\ 2b & 2c & 0 & 2a \\ 2c & 2b & 2a & 0 \end{vmatrix}.$$

6. Show that

$$\begin{vmatrix} bc & -ac & -ab \\ b^2 - c^2 & a^2 + 2ac & -a^2 - 2ab \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \frac{1}{4} abc (a+b+c)^3.$$

7. If $x = a(y + z)$, $y = b(z + x)$, $z = c(x + y)$, show that

$$ab + bc + ac + 2abc = 1.$$

8. If $x_1 - x_2 : x_2 - x_3 :: y_2 - y_3 : y_3 - y_1$, show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

9. If $\frac{a_1 - a_2}{x} = \frac{b_1 - b_2}{y}$, then

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ x & y & 0 \end{vmatrix} = 0.$$

10. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1d_1 + b_1e_1 + c_1f_1 & a_1d_2 + b_1e_2 + c_1f_2 & a_1d_3 + b_1e_3 + c_1f_3 \\ a_2d_1 + b_2e_1 + c_2f_1 & a_2d_2 + b_2e_2 + c_2f_2 & a_2d_3 + b_2e_3 + c_2f_3 \\ a_3d_1 + b_3e_1 + c_3f_1 & a_3d_2 + b_3e_2 + c_3f_2 & a_3d_3 + b_3e_3 + c_3f_3 \end{vmatrix}.$$

11. Show that

$$\frac{1}{x} \begin{vmatrix} y & x & 0 \\ z & 0 & x \\ -a & -b & -c \end{vmatrix} + \frac{1}{y} \begin{vmatrix} 0 & z & y \\ y & x & 0 \\ -a & -b & -c \end{vmatrix}$$

$$+ \frac{1}{z} \begin{vmatrix} z & 0 & y \\ 0 & z & x \\ -b & -a & -c \end{vmatrix} = ax + by + cz.$$

12. Show that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}^2 = \begin{vmatrix} a^2 + b^2 + c^2 & ad + be + cf & ag + bh + ck \\ ad + be + cf & d^2 + e^2 + f^2 & dg + eh + fk \\ ag + bh + ck & dg + eh + fk & g^2 + h^2 + k^2 \end{vmatrix}.$$

13. If

$$\begin{array}{rcl} (a + \beta + \gamma) x & - & \gamma y & - & \beta z = e, \\ - & \gamma x + (b + \gamma + \alpha) y & & - & \alpha z = 0, \\ - & \beta x & - & \alpha y + (c + \alpha + \beta) z = 0, \end{array}$$

and

$$d = \begin{vmatrix} a + \beta + \gamma, & -\gamma, & -\beta \\ -\gamma, & b + \gamma + \alpha, & -\alpha \\ -\beta, & -\alpha, & c + \alpha + \beta \end{vmatrix};$$

show that

$$x = \frac{e}{d} \{ (b + \gamma)(c + \beta) + \alpha(b + c + \beta + \gamma) \}.$$

Miscellaneous Examples.

1. Show that abc is $> (a + b - c)(a + c - b)(b + c - a)$, except when $a = b = c$.

2. If the number n be divided into any two parts, the difference of the squares of the parts is n times the difference of the parts.

3. Find to five places of decimals the value of x from the equation

$$\frac{2}{3 + x} + \frac{1}{5 - x} = 4.$$

4. Show that

$$\begin{aligned} & (a^2 + b^2)(c^2 + d^2)(e^2 + f^2) \\ &= (adf + bcf + bed - ace)^2 + (bce + aed + acf - bdf)^2. \end{aligned}$$

5. Prove that

$$\begin{aligned} & 25 \{ (y - z)^7 + (z - x)^7 + (x - y)^7 \} \\ & \times \{ (y - z)^3 + (z - x)^3 + (x - y)^3 \} \\ &= 21 \{ (y - z)^5 + (z - x)^5 + (x - y)^5 \}^2. \end{aligned}$$

6. Clear of surds the equations

$$\begin{aligned} \sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} &= 0, \text{ and} \\ \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{u} &= 0. \end{aligned}$$

7. Find x and y from the equations

$$\begin{aligned} x + \sqrt{\frac{x}{y}} &= \frac{42}{y}, \text{ and} \\ 2x^2y + 3x\sqrt{y} &= 324. \end{aligned}$$

8. Prove that

$$1 - n^2 + \frac{n^2(n^2 - 1^2)}{1^2 \cdot 2^2} + \frac{n^2(n^2 - 1^2)(n^2 - 2^2)}{1^2 \cdot 2^2 \cdot 3^2} + \dots = 0,$$

when n is a positive integer.

9. Show that the sum of the squares of the coefficients of

$$(1 - x + x^2)^n \\ = \frac{2n}{(n)^2} \left\{ 1 + \frac{n^2}{2} + \frac{n^2(n-1)^2}{4} + \dots \right\}.$$

10. For what values of x is $x + \frac{3}{x} >$ or < 4 ; and find the least value of $\frac{(1+x)(2+x)}{3+x}$.

11. If α, β be the roots of the quadratic $ax^2 + bx + c = 0$, show that the equation whose roots are $\frac{\alpha^4}{\beta}$ and $\frac{\beta^4}{\alpha}$ is

$$\alpha^4 c x^2 - b(b^4 - 5ab^2c + 5a^2c^2)x + ac^4 = 0.$$

12. What is the interpretation of the expression $x \sqrt[n]{n}$?

13. Show that unless $a = b = c = \dots$, then

$$\frac{bcd \dots}{(b-a)(c-a)(d-a) \dots} + \frac{acd \dots}{(a-b)(c-b)(d-b) \dots} \\ + \dots = 1.$$

14. Sum the series $1^p + 2^p + 3^p + \dots + n^p$.

15. Show that

$$\frac{1}{6} \{ (x-y)^5 + (y-z)^5 + (z-x)^5 \} \\ = \frac{1}{6} \{ (x-y)^3 + (y-z)^3 + (z-x)^3 \} \{ (x-y)^2 + (y-z)^2 + (z-x)^2 \}.$$

16. Show that

$$\frac{a^{n+2}}{(a-b)(c-a)} + \frac{b^{n+2}}{(a-b)(b-c)} + \frac{c^{n+2}}{(b-c)(c-a)}$$

is always equal to the sum of the homogeneous products of n dimensions of a, b, c .

17. The integral part of $(1 + \sqrt{3})^{2m+1}$ is divisible by 2^{m+1} .

18. If $E^n = y + \sqrt{1 + y^2}$, then $y = \frac{E^n - E^{-n}}{2}$.

19. Resolve into factors the expression

$$1 + a_0 + (1 + a_0) a_1 + (1 + a_0) (1 + a_1) a_2 \dots + \{(1 + a_0) (1 + a_1) \dots (1 + a_{n-1}) a_n\}.$$

20. Show that

$$\sqrt{\frac{b}{2} + \sqrt{ab - a^2}} + \sqrt{\frac{b}{2} - \sqrt{ab - a^2}} = \sqrt{2} a.$$

21. Show that $\left(\frac{-1 + \sqrt{-3}}{2}\right)^n + \left(\frac{-1 - \sqrt{-3}}{2}\right)^n = 2$, if n

be a multiple of 3, and is equal to -1 if n be any other integer.

22. Show how to sum the series $a_n x^n + a_{n+n} x^{n+n} + a_{n+2n} x^{n+2n} + \&c.$, when the sum of $a_0 + a_1 x + a_2 x^2 + \dots$ is known.

23. If $(a + b\sqrt{-1})^{\frac{1}{n}} = c + d\sqrt{-1}$, then

$$(a - b\sqrt{-1})^{\frac{1}{n}} = c - d\sqrt{-1}.$$

24. If $a_1, a_2, a_3, \&c., a_n$ be unequal, and $m < n - 1$, then

$$\frac{a_1^m}{(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n)} + \frac{a_2^m}{(a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n)} + \dots + \frac{a_n^m}{(a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})} = 0.$$

25. If $\frac{ad - bc}{a - b - c + d} = \frac{ac - bd}{a - b - d + c}$, then each of them

$$= \frac{1}{4} (a + b + c + d).$$

26. Resolve into partial fractions

$$\frac{x^2 + mx + n}{(x + a)(x + b)(x + c)}.$$

27. Find the number of homogeneous products of n quantities $a, b, c, \&c.$, of r dimensions. Hence find the number of terms in the expansion of $(a + b + c + \&c.)^n$, n being integral.

28. Find a factor which will rationalize $a^{\frac{1}{2}} - b^{\frac{1}{2}}$, and extract the square root of $37 - 20\sqrt{3}$.

29. If $ax^3 = by^3 = cz^3$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$, show that $ax^3 + by^3 + cz^3 = (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 a^{\frac{2}{3}}$.

30. If a debt a at compound interest is discharged in n years by annual payments of $\frac{a}{m}$, show that

$$(1 + r)^n (1 - mr) = 1.$$

31. If $x = h\sqrt{s^2 + .0075a^2}$, find the value of x when $h = 28.4$ metres, $s = 3.2$ metres, $a = 5.2$ kilometres. Of what dimension is x ?

32. Show that $\frac{p+q+r+s}{a+b+c+d}$ lies between the greatest and least of the fractions $\frac{p}{a}, \frac{q}{b}, \frac{r}{c}, \frac{s}{d}$, but differs from the average value of those quantities.

33. If $u_r = \frac{x \cdot \overline{x+1} \dots \overline{x+r-1}}{\overline{1}^r}$, show that $u_1 + u_2 + \dots + u_n = \frac{x + \overline{1} \cdot \overline{x+2} \dots \overline{x+n}}{\overline{1}^n} - 1$.

34. Show that the minimum value of

$$\frac{(a+x)(b+x)}{x} \text{ is } (\sqrt{a} + \sqrt{b})^2.$$

35. If $10^x = 2$, find x as a continued fraction.

36. Calculate T from the following formula:

$$T = \frac{2A}{mS\sqrt{2g}}\sqrt{H},$$

when $A = 3503.6$ sq. ft., $S = 13.532$ sq. ft., $m = .548$, $g = 32.1998$ ft., $H = 6.3945$ ft.

37. Reduce to its simplest form

$$\frac{(yz - ax)^2 - (ac - y^2)(ab - z^2)}{(bc - x^2)(yz - ax) - (xz - by)(xy - cz)}$$

38. If p and q are each less than 1, show that

$$\frac{\log(1-p)}{\log(1-q)} \text{ is } > \frac{p-pq}{q}, \text{ and } < \frac{p}{q-pq}.$$

39. If $x = \frac{2ab + b^2}{a^2 + ab + b^2}$ and $y = \frac{a^2 - b^2}{a^2 + ab + b^2}$, show that $x^2 + y = y^2 + x$.

40. Prove that

$$\log_e x = \frac{1}{n} \cdot {}^nS_m \left(\frac{x^m - 1}{m(x^n + 1)^m} \right),$$

where ${}^nS_m f(m)$ denotes the sum of n terms of the series whose m th term is $f(m)$.

41. Show that the integral part of $(5 + \sqrt{24})^n$ is odd if n be a positive integer.

42. If $\frac{A}{x} = \frac{B}{y} = \frac{C}{z}$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then

$$\frac{A^2}{a^2} + \frac{B^2}{b^2} + \frac{C^2}{c^2} = \frac{A^2 + B^2 + C^2}{x^2 + y^2 + z^2}.$$

43. Write down the coefficient of t^n in the expression

$$1 - \frac{1}{2}(e^t - 1) + \frac{1}{3}(e^t - 1)^2 - \&c. \\ + \frac{1}{2n+1}(e^t - 1)^{2n} - \&c.$$

44. On a sum of money borrowed, interest is paid at the rate of 5 per cent. After a time £600 of the loan is paid off, and the interest on the remainder reduced to 4 per cent., and the yearly interest is now lessened one-third. What was the sum borrowed?

45. If $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$, find x in terms of y by the method of indeterminate coefficients.

46. Show how the solution of the equation $ax + by = c$ in positive integers may be obtained by the properties of

continued and converging fractions. Solve $17x + 19y = 250$, in positive integers.

47. In the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$, find the value of $a^2\beta + a^2\gamma + \beta^2a + \beta^2\gamma + \&c.$, where a, β, γ are the roots of the equation.

$$48. \text{ If } \frac{a^3}{(a-\beta)(a-\gamma)} + \frac{\beta^3}{(\beta-a)(\beta-\gamma)} + \frac{\gamma^3}{(\gamma-a)(\gamma-\beta)} = L,$$

$$\frac{a^4}{(a-\beta)(a-\gamma)} + \frac{\beta^4}{(\beta-a)(\beta-\gamma)} + \frac{\gamma^4}{(\gamma-a)(\gamma-\beta)} = M,$$

$$\frac{a^5}{(a-\beta)(a-\gamma)} + \frac{\beta^5}{(\beta-a)(\beta-\gamma)} + \frac{\gamma^5}{(\gamma-a)(\gamma-\beta)} = N,$$

show that $LN - M^2 = a^2\beta^2 + a^2\gamma^2 + \beta^2\gamma^2 - a\beta\gamma(a + \beta + \gamma).$

49. Sum to n terms, and to infinity

$$\frac{x}{(1+x)(1+ax)} + \frac{ax}{(1+ax)(1+a^2x)}$$

$$+ \frac{a^2x}{(1+a^2x)(1+a^3x)} + \&c.$$

50. Show that the result of eliminating x, y, z from the equations

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0,$$

is the same as the result of elimination from

$$a_1x + a_2y + a_3z = 0,$$

$$b_1x + b_2y + b_3z = 0,$$

$$c_1x + c_2y + c_3z = 0.$$

ANSWERS.

I.—PAGE 13.

1. 8, 27, 6, - 16. 2. 6, 2, 4, 0. 3. 5, 11, - 7.
 4. 11, 16. 5. - 2, - 15. 6. 8. 7. - 5, - 7.
 8. - 11, - 3. 9. 7, - 69. 10. 72, - 68.
 11. - 2, - 8. 12. - 10, 32.

II.—PAGE 18.

1. 51, 6, 3. 2. 125, 35. 3. - 513, - 65.
 4. - 1224, 30. 5. 0, 2. 6. 1, - 27. 7. 2, - 2.
 8. 1591, 0. 9. 144. 10. 1521.
 11. 145. 12. $-\frac{37}{\sqrt[3]{1029}}$.

III.—PAGE 21.

1. $10a + 3b$. 2. $11a^2$. 3. $12b + 8c$. 4. 0.
 5. $8a^2 + ab$. 6. $3x^4 - x^2 - 15x - 2$. 7. $4ab - 4$.
 8. $x^3 + y^3 + z^3 - 3xyz$. 9. $x^4 + x^2y^2 + y^4$.
 10. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 - a^2c - ac^2 - b^2c - bc^2 - 2abc$.
 11. $x^4 + y^4 + z^4 - 4x^2y + 4x^2z - 4xy^2 - 4y^2z + 4xz^2 - 4yz^2 + 6x^2y^2 + 6x^2z^2 + 6y^2z^2 - 12x^2yz + 12xy^2z - 12xyz^2$. 12. 0.

IV.—PAGE 22.

1. $4a + 2b + 5c$. 2. $-4x + 2y - 6z$.
 3. $a^2 - 3ab - b^2 - 5a - 7b - 8$.

4. $2a^4 - 2a^3x^3 + x^4$. 5. $4a^4 + 8a^2b^2 + 4b^4$.
 6. $-3x^3 - 3z^3 - 3x^2z - 3xz^2$.
 7. $x^4 - ax^3 - 9a^2x^2 - 3a^3x - 2a^4$. 8. 0.
 9. $-a - b - c - d + e + f + g + h$.
 10. $2x^4 - 8x^2y + 12x^2y^2 - 8xy^3 + 2y^4$.
 11. $x^4 - 2x^2y^2 + y^4$.
 12. $a^3 + b^3 - 2c^3 + 2ab - 2ac - 2bc$.

V.—PAGE 24.

1. $-x + 8y + z$. 2. $a^3 - b^3$. 3. $13 - 6x$.
 4. $5x^3 - 3x - 7$.
 5. $(a + b) + (c - d), a - (b - c + d),$
 $\{a - (b - c)\} - d$.
 6. $-(6a - 7b) - (3c - 5d), -6a + (7b - 3c + 5d),$
 $- \{6a - (7b - 3c)\} + 5d$.
 7. $-(4x^3 - 12x^2y) - (12xy^2 - 4y^3),$
 $-4x^3 + (12x^2y - 12xy^2 + 4y^3),$
 $- \{4x^3 - (12x^2y - 12xy^2)\} + 4y^3$.
 8. $(a^3 - b^3) - (c^3 - 3abc), a^3 - (b^3 + c^3 - 3abc),$
 $\{a^3 - (b^3 + c^3)\} + 3abc$.
 9. $-(a - b + d)x^2 - (a - b - 3c)xy$
 $-(2b - c - e - f)y^2$.
 10. $(a - b + c - d)x - (c - d + e - f)y$
 $-(e - f + g - h)z$.
 11. $(a - b - d)x - (a - 7b - c)y - (b - c + d)z$.
 12. $(z - x)a^3 + (x - y)ab + (y - z)b^3$.

VI.—PAGE 27.

1. $12a^3 - ab - 6b^3, 18x^3 - 9xy - 35y^3$.
 2. $x^3 - 6xy^2 + 4y^3, 30x^3 + 49x^2y + 9xy^2 - 4y^3$.
 3. $a^4 - 2a^2b^2 + b^4, a^4 - b^4$. 4. $x^4 - y^4, x^3 - y^3$.
 5. $a^3 + b^3 + c^3 - 3abc$. 6. $x^3 - y^3$.
 7. $a^3 + 3a^2b + 3ab^2 + b^3$.
 8. $5a^5 + 5a^4 - 405a - 405$.

9. $-7x^3 + 17x^2 - 5x - 2, a^3 - x^3.$
10. $a^3 + 2a^2b^3 + 3a^2b^4 + 2a^2b^5 + b^5.$
11. $af + (ac + bf)x + (b + f)cx + (c^2 + df)x^2 + cdx^3.$
12. $x^3 + (p - a)x^2 + (q - ap)x - aq.$
13. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$

VII.—PAGE 32.

1. $x^3 - 2xy + y^3, 9a^2 - 30ab + 25b^2, 16c^4 + 8c^2d^2 + d^4,$
 $9x^4 - 12x^2y^2 + 4y^4.$
2. $a^3 - b^3.$ 3. $m^2x^3 - n^2y^3, 50a^4 - 18b^4.$
4. $(a + c)^3 - (b + d)^3, (a + b)^3 - (c + d)^3.$
6. $x^2 + 4x - 5, x^2 + 7x + 10, x^2 + 2x - 15, x^2 - 25.$
7. $x^4 - 37x^2 - 24x + 180.$
8. $\frac{1}{18}\{4a^2b^3 - (a^3 - b^3 - c^3)^2\}.$
9. $4(a^3 + b^3 + c^3 + d^3).$

VIII.—PAGE 38.

1. $4a^2 - ab + 2b^2, xy - 4y^2.$
2. $2a^4 - 5a^3b + 3a^2b^2, 2x^2 + 3xy + 2y^2.$
3. $ax^{m-n} + bx^{-n}y^n + cx^{-(m+n)}y^{2n},$
 $ax^{m-n} + by^n + cx^{n-m}y^{2n}.$
4. $6x + 4y, 5x - 3y.$ 5. $1 + x + x^2.$
6. $-3x - 4, -7x + 3.$
7. $x^3 - 3x^2y + 5xy^2 + 27y^3 + \frac{72y^4}{x - 3y},$
 $x^3 - x^2y - 3xy^2 + 15y^3 - \frac{24y^4}{x + y}.$
8. $x^4 + x^3y + x^2y^2 + xy^3 + y^4, x^4 - x^3y + x^2y^2 - xy^3 + y^4.$
9. $ax^m + by^n.$ 10. $a^3 + a^2b + ab^2 + b^3.$
11. $b + c, a + b.$ 12. $a + bx + cx^2.$
13. $a^4 - (p - 1)a^3 - (p - q - 1)a^2 - (p - 1)a + 1.$
14. The given expression is—
 $(a + b + c - d)(a + b - c + d)(a - b + c + d)(a - b - c - d).$

16. $-x^4y^2 - x^2y^4, x + x^{-1}$.
 17. $(x+y)^2 + (x+y)^2z - (x+y)z^2 - z^3$.
 20. $(a-c)^2 - 2(a-c)(b-d) + (b-d)^2$.

IX.—PAGE 45.

1. $(x+3a)(x-3a), (4y^2+5z^2)(4y^2-5z^2),$
 $6(2a+3b)(2a-3b), (2x-3y)(4x^2+6xy+9y^2).$
2. $x(x-y)(x^2+xy+y^2),$
 $(a-b)(a+b)(a^2+b^2)(a^4+b^4),$
 $xy(x+y)(x^2-xy+y^2),$
 $2xy^2z(x+2z)(x-2z).$
3. $(a^2-2b^2)(a^2+2b^2), (x^2+xy+y^2)(x^2-xy+y^2),$
 $(a+b)^2(a-b)^2(a+b+c)(a+b-c).$
4. $(a+b+c+d)(a+b-c-d),$
 $(a+b-c+d)(a-b+c+d),$
 $(a+b-c)(a-b+c).$
5. $5(2x+9), 3(2x+7), (3a+b-c)(a+b+c).$
6. $(x^2+xy+y^2)\{(x-y)(x^2-y^2)+4x^2y^2(x^2-xy-y^2)\},$
 $(x^2+xy+y^2)(x^2-xy+y^2)(x+y)^2(x-y)^2.$
7. $(x-10)(x+7), (x+1)(x+10),$
 $(a-7b)(a-8b), (x-16)(x+12).$
8. $(ax+7by)(ax-6by), 3a(x-10)(x+2),$
 $ac(c-8)(c+3).$
9. $(3x+5)(2x-7), (2x+9)(4x-15),$
 $3(2x+3)(3x-8), (4x-7)(5x+6).$
10. $xy(3x+y)(x+3y), x(5x+3a)(4x+5b),$
 $(mx+p)(nx+q).$
11. $x^3+2x^2+4x+8, 3x^4-6x^3+12x^2-24x+48,$
 $x^4+3x^2+9.$
12. $(a+b)^2-(a+b)(c+d)+(c+d)^2, a-b+c+d.$
13. $a^4+pa^3+qa^2+ra+s, 2a^4.$ 14. $-102, 17.$

X.—PAGE 48.

1. $a^4b^6, -27a^3b^3, 16a^2b^2c^2, -x^2y^2z^2.$

2. $a^4 + 12 a^2 b + 54 a^2 b^2 + 108 ab^3 + 81 b^4$,
 $16 a^4 + 32 a^2 b + 24 a^2 b^2 + 8 ab^3 + b^4$,
 $a^5 - 5 a^4 b + 10 a^2 b^2 - 10 a^2 b^3 + 5 ab^4 - b^5$,
 $27 a^3 - 108 a^2 b + 144 ab^2 - 64 b^3$.
3. $16 m^4 + 32 m^3 + 24 m^2 + 8 m + 1$,
 $125 x^3 + 150 x^2 + 60 x + 8$,
 $81 a^4 - 432 a^3 c + 864 a^2 c^2 - 768 ac^3 + 256 c^4$,
 $- a^3 - 3 a^2 b - 3 ab^2 - b^3$.
4. $x^4 + 2 x^3 + 3 x^2 + 2 x + 1$,
 $9 a^3 + b^3 + 16 c^2 + a^2 - 6 ab + 24 ac - 6 ad - 8 bc$
 $+ 2 bd - 8 cd$,
 $a^2 + 4 b^2 + c^2 + 4 ab - 2 ac - 4 bc$,
 $9 (a^3 + b^3 + c^2 + 2 ab + 2 ac + 2 bc)$.
5. $1 + 3x - 5x^2 + 3x^3 - x^4$, $(ax + by)^3 + 3(ax + by)cx + 3cx$.
6. $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$,
 $(1 + x)^4 + 4(1 + x)^3 x^2 + 6(1 + x)^2 x^4 + 4(1 + x)x^6 + x^8$,
 $(a + bx)^4 + 4(a + bx)^3 cx^2 + 6cx$.
7. $a^5 x^5 - 8 a^4 b x^4 y + 28 a^3 b^2 x^3 y^2 - 56 a^2 b^3 x^2 y^3 + 70 a b^4 x y^4$
 $- 56 a^3 b^2 x^2 y^5 + 28 a^2 b^3 x^2 y^6 - 8 a b^4 x y^7 + b^5 y^8$,
 $729 x^6 - 243 x^4 y^2 + 27 x^2 y^4 - y^6$,
 $x^5 - 3 x^3 y^2 + 3 x^2 y^3 - y^5$.
8. $x^{12} - 2 x^6 y^6 + y^{12}$,
 $a^{12} - 6 a^{10} b^2 + 15 a^8 b^4 - 20 a^6 b^6 + 15 a^4 b^8 - 6 a^2 b^{10} + b^{12}$.
9. $a^3 + 3 a^2 b + 3 ab^2 + b^3$. 10. $(a - c)^3$
11. $2(1 + 3x^2)^4$. 12. $81 x^4 y^4 (x + y)$.

XI.—PAGE 62.

1. $2xy^2z^2$, $4a^2y^2$, $x^2 + a^2$. 2. $2x^3 - 3x^2y + 5xy^2$.
3. $5a^2 - 3ab + b^2$. 4. $1 - 2x + 3x^2 - 4x^3$.
5. $a + bx + cx^2 + dx^3$. 6. $a^2x^n - 3ax^{n-1} + 4x^{n-2}$.
7. $x + x^{-1}$, $ax^{-1} - a^{-1}x$ 8. $3x^n - \frac{a^2}{2} 5a$.
9. 36, 79, 207, 289. 10. 103·2, ·024, ·3, $\frac{1}{4}$.
11. 4·123, 1·224, ·618, 3·732.
12. ·0203, 4·6, 3·5036. 13. $2ab^2y^4$, $5x^4y$, $a + 2b$.

- | | |
|--|------------------------------|
| 14. $x^4 + 3x^2 - 7$. | 15. $x + y - c$. |
| 16. $x + x^{-1}, xy^{-1} + 1$. | 17. 18018, 11·11. |
| 18. 2·73, $\cdot\dot{3}$, $\frac{1}{3}$. | 19. $\cdot 5$, 2599. |
| 20. $\sqrt[3]{5} = 1\cdot709$. | 21. 7·6457, 50·2487. |
| 22. 4·4142, 3·7320. | 23. $\cdot 25$. 24. 0. |

XII.—PAGE 69.

- | | | |
|-----------------------|-----------------------|-----------------------------|
| 1. $x - 3$. | 2. $x^2 + 4x + 3$. | 3. $x + 3$. |
| 4. $x + a$. | 5. $a^2 + ab + b^2$. | 6. $x - 1$. |
| 7. $x - 3$. | 8. $3x - 2$. | 9. $12x^2 + 5x$. |
| 10. $3a + 2b$. | 11. $x - 1$. | 12. $2x^2 - 4x^2 + x - 1$. |
| 13. $2a + 3b + c$. | 14. $x^2 + 1$. | 15. $x - 1$. |
| 16. $2x^2 - 3x + 9$. | 17. 1. | 18. $a + b + c$. |

XIII.—PAGE 72.

- | | |
|--|--------------------------------|
| 1. $12a^2x^2y^3$. | 2. $60a^2b^2c^2$. |
| 3. $(a - b)(b - c)(c - a)$. | 4. $a^2x(x^2 - a^2)$. |
| 5. $(x + 1)(x + 2)(x + 3)$. | 6. $(x - 6)(x + 5)(x - 5)$. |
| 7. $(2x + 7)(3x + 8)(4x + 5)$. | |
| 8. $210(x^2 + 1)(x^3 + 1)$. | 9. $a^3(x^4 + a^2x^2 + a^4)$. |
| 10. $(x + a)(x + b)(x + c)$. | 11. $1 - x^{16}$. |
| 12. $(x + 1)(x + 2)(x + 3)(x - 5)(x - 6)(x + 5)$. | |
| 13. $a(a - b)^3(a^2 - ab + b^2)$. | |
| 14. $(x - 1)(x + 1)(x^2 + 1)(6x^3 + 5x^2 + 2x - 1)$. | |
| 15. $(a^2 - b^2)^4$. | |
| 16. $2x(x^2 - 1)(3x^2 + 3x - 1)(2x^2 + 2x - 5)(2x^2 - 2x + 5)$. | |
| 17. $3(x - 2)^2(x + 4)(5x^2 + 10x^2 + 20x + 18)$. | |
| 18. $a^2b^2(a + b)(a - b)$. | 19. $15(3x - 10)(x^4 - 16)$. |
| 20. $(x^2 - y^2)^2(x^2 + y^2)(x^4 + y^4)$. | 21. $x^5 - a^5$. |
| 22. $(a + b + c + d)(a + b - c - d)(a - b - c + d)$. | |
| 23. $(a + b + c)(a^3 + b^3 + c^3 - 3abc)$. | |
| 24. $(a + b + c + d)(b + c + d - a)(a + c + d - b)$
$(a + b + d - c)(a + b + c - d)(a + b - c - d)$. | |

XIV.—PAGE 75.

1. $\frac{x-1}{x+6}, \frac{x^2+x-2}{x^2+5x+5}$
2. $\frac{3x+7}{7x+2}, \frac{2x^2+2x+9}{5x^2+2x-2}$
3. $\frac{(a-b)(a+b)^2}{a^2-3ab+b^2}, \frac{4a^2+b^2}{a+3b}$
4. $\frac{x(y-z)-y^2}{x(y+z)-y^2}, \frac{1}{x^2-y^2}$
5. $\frac{2a}{a^2-b^2}, \frac{2b}{a^2-b^2}$
6. 0.
7. $\frac{x^2-x^2+3x-1}{2(x-1)(x^2+1)^2}$
8. $\frac{10}{(x+1)(x+2)^2(x^2+1)}$
9. $\frac{8x-20}{(x+1)^2(x+3)}$
10. $\frac{8x^2+8}{(x-1)^4(x^2+1)}$
11. 0.
12. 0.
13. 1.
14. $a+b+c$.
15. 1.
16. 0.
17. $-\frac{16a^5x}{(a^4-x^4)^2}$
18. $\frac{a^2-ab+b^2}{a^2+ab-b^2}$
19. 1.
20. 1.
21. $\frac{8}{(x-y)(y-z)(x-z)}$
22. $\frac{1}{x^2}$
23. $\frac{1}{(x-a)(x+b)(x+c)}$
24. $\frac{5a^2}{x} - 4a + 3x$
25. 1.
26. $\frac{4a^2b^2}{(a^2-b^2)^2}$
27. $\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{a-b}\right)^2$
28. $\frac{a^2(x-y)}{(a+x)(a+y)^2}$
29. 1.
30. $\frac{1}{x^2}$

XV.—PAGE 82.

- | | | | |
|-------|-------|------------|-------|
| 1. 3. | 2. 4. | 3. $a+b$. | 4. 5. |
| 5. 6. | 6. 3. | 7. 5. | 8. 2. |

9. c . 10. $a + b$. 11. $a + b + c$. 12. $\frac{a^2 + ab + b^2}{b}$.
13. abc . 14. $-(a + b)$. 15. abc . 16. $\frac{2a(2b^2 - 5)}{4a - 3b}$.
17. 2. 18. 8. 19. $\frac{1}{4}$. 20. $\frac{cd - ab}{a + b - c - d}$.
21. $\frac{a^2 + ab + b^2}{a + b}$. 22. $\frac{1}{abc}$. 23. $a + b + c$.
24. $\frac{1}{a + b + c}$.

XVI.—PAGE 87.

1. 6. 2. $\frac{ab - cd}{a + b - c - d}$. 3. 40.
4. $\frac{1}{2} \frac{1}{3} \frac{1}{4}$. 5. $-\frac{1}{12}$. 6. $-\frac{1}{3}$.
7. $1\frac{1}{2}$. 8. 3. 9. $-1\frac{1}{2}$.
10. 8. 11. $1\frac{1}{2}$. 12. $\frac{3 - m - n}{2a}$.
13. $-\frac{a^2 + b^2}{a + b}$. 14. $2\left(\frac{1 - b}{1 + b}\right)^2 - \left(\frac{1 - b}{1 + b}\right)^4$.
15. 3. 16. - 8. 17. 1. 18. $\frac{b^2 + 1}{2ab}$.
19. $(\sqrt{a} + \sqrt{b})^2$. 20. $-a$. 21. $a^2 + 2a$.
22. $\frac{1}{4}$. 23. $-2\frac{1}{2}$. 24. $\frac{af(bc - ad) + (gf - ae)cd}{a^2(bc - ad) - (gf - ae)c^2}$.

XVII.—PAGE 92.

1. 10. 2. 45, 25. 3. £360, £240, £120.
4. 20, 15 miles per hour. 5. 24.
6. 36s., 48s. 7. 12. 8. 36s. 9. 30s.
10. 13. 11. 10. 12. 24 lbs., 40 lbs.

13. 13. 14. 500. 15. 15s. 16. $46\frac{2}{3}$.
 17. 18 miles from A. 18. $2\frac{1}{4}$ hr.
 19. 15 per cent. 20. 28 miles. 21. 10.
 22. $4\frac{1}{2}$, 4 miles per hour. 23. 12 miles.
 24. 20 miles per hour in same direction.
 25. 120. 26. 6s., 4s. 27. 2, 1, $\frac{1}{2}$, 6 hours.
 28. 98 per cent. 29. £1,000. 30. $38\frac{1}{2}$ miles per hour.
 31. 2 inches. 32. 16 vols. of hydrogen, 8 of oxygen.
 33. $\frac{1 - 100c}{b - c}$, $\frac{100b - 1}{b - c}$. 34. $\frac{ab'c - a'bc'}{cc'(a'b - ab')}$.
 35. -40° . 36. $\frac{a^2b}{c - ab}$.

XVIII.—PAGE 99.

1. 3, 4. 2. 5, 2. 3. 3, 7. 4. 4, 5.
 5. 12, 8. 6. 2, 6. 7. 1, 2, 3. 8. 1, 2, 3.
 9. 4, 1, 2 10. $\frac{b_1c - bc_1}{ab_1 - a_1b}$, $\frac{ac_1 - a_1c}{ab_1 - a_1b}$.
 11. $x = \frac{a + c - b}{2}$, &c. 12. $x = \frac{d + f - e}{2a}$.
 13. 4, 5. 14. 3, 8. 15. 4, 8. 16. 3, 4.
 17. 3, 5. 18. 8, 1. 19. $x = \frac{2}{a + c - b}$, &c.
 20. 2, 3, 4, 1. 21. 3, 4, 5. 22. $x = \frac{a + c + d - 2b}{3}$, &c.
 23. $x = \frac{m(a + c - b)}{2}$, &c. 30. $x = y = z = L$

XIX.—PAGE 102.

1. 12, 8. 2. 37. 3. 6 lbs., 5 stones.
 4. $\frac{1}{2}$. 5. 17 horses, 24 cows.

6. 6s., 3s. 7. 220 and £2. 14s.
 8. 200, 300, 400. 9. 48, 23, 18. 10. 5, 3, $4\frac{1}{2}$.
 11. 4, 6, 8 hours. 12. 6, 10, 18. 13. 71, 12, 3.
 14. 4, 3 miles per hour. 15. 16, 32, 48 miles per hour.
 16. $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$
 $a_3x + b_3y + c_3z = d_3$ from which x, y, z
 20. $30\frac{9}{20}$, $152\frac{1}{4}$.

XX.—PAGE 109.

1. ± 3 . 2. ± 4 . 3. ± 5 .
 4. $\pm \frac{1}{2}$. 5. ± 2 . 6. $\pm (a + b)$.
 7. ± 1 . 8. $\pm \sqrt{a^2 - b^2}$. 9. $\pm \sqrt{2ab - b^2}$.
 10. $\pm \frac{2}{3a} \sqrt{3}$. 11. $\pm a \sqrt{\frac{3b^2 + 1}{b^3 + 3b}}$.
 12. $\pm \frac{a + 1}{2\sqrt{a}}$. 13. 2, 3. 14. 9, - 8.
 15. 1, - $\frac{1}{2}$. 16. 4, - $\frac{1}{2}$. 17. - $\frac{1}{2}$, - $\frac{1}{2}$.
 18. - 5, - $\frac{1}{2}$. 19. $\frac{1}{2}(b \pm \sqrt{b^2 - 4ac})$.
 20. 5, $3\frac{1}{2}$. 21. 7, - 6. 22. 6, - $\frac{1}{2}$.
 23. 1, $\frac{ad - (a + b)c}{bc}$. 24. b, a .
 25. $a + b, \frac{ab}{a + b}$. 26. a, b . 27. $\frac{c}{a - b}, \frac{d}{a - b}$.
 28. $\frac{a^2 + ab + b^2}{a - b}, \frac{a - b}{a^2 - ab + b^2}$. 29. 4, 10 .
 30. 5, - $7\frac{1}{2}$. 31. 3, - 5. 32. 2, - $3\frac{1}{2}$.
 33. 2, - $1\frac{1}{2}$. 34. 5, $\frac{1}{2}$. 35. 8, - $\frac{9}{4}$.
 36. 3, - $2\frac{1}{2}$. 37. 2, $2\frac{1}{2}$. 38. 4, - $\frac{1}{2}$.
 39. 15, - 1. 40. 2, 0. 41. 0, $2ab$.
 42. $a, 3\frac{1}{2}a$. 43. 0, $\frac{a^2 + b^2 + 2b\sqrt{ab}}{(b - a)b}$.

$$44. 1, \frac{a^2b^2(a^2 - b^2)^2}{\{(a^2 + b^2)^2 + 2ab(a + b)^2\}^2 - 4a^2b^2}$$

$$45. a + b + c, -\frac{a^2 + b^2}{a - b}.$$

XXI.—PAGE 115.

1. ± 3 . 2. $2, -\sqrt[3]{\frac{1}{2}}$. 3. $\pm 5, \pm \frac{1}{\sqrt{5}}$.
 4. $2, -5$. 5. $\pm \frac{1}{2}$. 6. $\sqrt[3]{2}, \sqrt[3]{3}$. 7. $4, 69$.
 8. $\pm 4, \pm \sqrt{7}$. 9. $3, -4, 3 \pm \sqrt{21}$. 10. $\frac{1}{2}, 4$.
 11. $-2, -2 \pm \sqrt{3}$. 12. $\frac{a}{2}$.
 13. $4, \frac{1}{2}, -\frac{1}{2}(-13 \pm \sqrt{145})$. 14. $0, b - a$.
 15. $\pm \frac{1}{2}a\sqrt{3}$. 16. $\frac{a}{2} \pm \sqrt{b - 2a\sqrt{b} + \frac{a^2}{2}}$.
 17. $0, \left(\frac{a \pm b}{a \mp b}\right)^{\frac{2}{p-2}}$. 18. $\pm \frac{a}{\sqrt{2}} \sqrt{\sqrt{5} - 1}$.
 19. $\left(\frac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2$. 20. $\pm 5, \pm \sqrt[3]{\frac{1}{2}}$.
 21. $2, -\frac{1}{2}, \frac{1}{2}(-11 \pm \sqrt{97})$. 22. $4, \frac{1}{2}$.
 23. $5, 3\frac{1}{2}, \frac{1}{2}(-7 \pm \sqrt{53})$. 24. $3a$.
 25. $3, -2$. 26. $4, -14\frac{1}{2}, \frac{1}{2}(-19 \pm \sqrt{741})$.
 27. $3, -2\frac{1}{2}, \frac{1}{2}$. 28. $\frac{1}{2}(1 \pm \sqrt{4a - 3})$.
 29. ± 1 , and $x^4 + x^2 + 1 = 0$. 30. $\pm \sqrt{\frac{a + b + c}{3}}$.
 31. $a, \frac{1}{2} \{ \sqrt{4c^2 - 3a^2 + 2ab + b^2} - (a - b) \}$.
 32. $x = m$ is one solution.
 33. $x = a + b + c$ is one solution.

$$34. x = \frac{a}{m}, \text{ and } a^2 + ax + x^2 = 0.$$

$$35. x = \frac{b+d}{a+c} \text{ is one solution.}$$

$$36. x^2 - x + 1 = 0 \text{ gives two solutions.}$$

XXII.—PAGE 119.

$$1. x = 2, 3, y = 3, 2. \quad 2. x = 5, -3, y = 3, -5.$$

$$3. x = 3, 4, y = 4, 3. \quad 4. x = 2, 4, y = 4, 3.$$

$$5. x = 5, -2, y = 2, -5. \quad 6. \pm 7, \pm 6.$$

$$7. \pm 6, \pm 3. \quad 8. 4, 2. \quad 9. \pm 2, \pm 7.$$

$$10. \pm 4, \pm 3. \quad 11. x = \pm 3, \frac{1}{\sqrt{3}}, y = \pm 2, -\frac{8}{\sqrt{3}}.$$

$$12. x = 9, 4, y = 4, 9. \quad 13. x = 2, 8, y = 8, 2.$$

$$14. x = 3, 2, y = 2, 3.$$

$$15. x = \frac{2}{a \pm \sqrt{2b^2 - a^2}}, y = \frac{2}{a \mp \sqrt{2b^2 - a^2}}$$

$$16. \pm \frac{(a+1)b}{\sqrt{2(a^2+1)}}, \pm \frac{(a-1)b}{2\sqrt{a^2+1}}.$$

$$17. \pm \sqrt{\frac{a+b^2}{2b}}, \pm \sqrt{\frac{a-b^2}{2b}}.$$

$$18. x = 2, 3, y = 3, 2.$$

$$19. x = 2, 3, y = 3, 2.$$

$$20. x = 2, 3, y = 3, 2.$$

$$21. x = 5, -3, y = 3, -5.$$

$$22. x = \frac{a + \sqrt{2b - a^2}}{\sqrt{4a^2 - 4b}}.$$

$$23. x = 2, 3, y = 3, 2.$$

$$24. x = \pm 3, \mp \frac{1}{8} \sqrt{\frac{3}{2}}, y = \pm 2, \pm \sqrt{\frac{3}{2}}$$

$$25. x = 4, 9, y = 9, 4. \quad 26. x = 4, 1, y = 8, 0.$$

$$27. x = 4, 3, y = 3, 4. \quad 28. \pm 3, \pm 1.$$

$$29. x = 0, 3, -\frac{1}{4}, y = 0, 2, \frac{1}{4}.$$

$$30. 0, -1. \quad 31. x = 4, \frac{1}{8}, y = 3, -\frac{1}{8}.$$

$$32. x = 5, 6, y = 4. \quad 33. x = 3, 1, y = 1, 0.$$

$$34. x = 3, -\frac{1}{2}, y = 1, -\frac{1}{2}.$$

$$35. x = 2, 7, y = 4, -14.$$

$$36. x + y = c, \text{ and } xy = bx + ay.$$

$$37. x = 0, a - b, y = 0, a - b.$$

$$38. \frac{1}{2} \pm 9\sqrt{\frac{1}{2}}, \frac{1}{2} \pm \sqrt{\frac{1}{2}}.$$

$$39. x = \sqrt{\frac{(a+c-b)(a+b-c)}{2(b+c-a)}}.$$

$$40. 1, 2, 3. \quad 41. \pm 1, 0, 0.$$

$$42. x = y = z = 0, \text{ also } \frac{1}{xy} = \frac{1}{2} \left(\frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a^2} \right),$$

$$\frac{1}{yz} = \frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{c^2} - \frac{1}{b^2} \right), \text{ \&c.}$$

$$43. \frac{a}{\sqrt[4]{a^3 + b^3 + c^3}}, \frac{b}{\sqrt[4]{a^3 + b^3 + c^3}}, \frac{c}{\sqrt[4]{a^3 + b^3 + c^3}}$$

$$44. x = y = z = u = a + b.$$

$$45. \frac{a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}}{(a+b+c)^{\frac{1}{2}}}, \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}}{(a+b+c)^{\frac{1}{2}}}, \text{ \&c.}$$

$$46. x = 6, 4, y = 4, 6, z = 5.$$

XXIII.—PAGE 123.

$$1. 9, -6. \quad 2. \text{Numerator } 2\frac{2}{3}, \text{ denominator } 3.$$

$$3. 35 \text{ or } 23. \quad 4. 10, -16. \quad 5. 78, -68 \text{ ft.}$$

$$6. 2s., 9s. \quad 7. 20 \text{ hours.} \quad 8. 48 \text{ shillings.}$$

$$9. 3 \text{ ft., } 2 \text{ ft.} \quad 10. 1, \frac{1}{2}. \quad 11. 3 \text{ ft., } 4 \text{ ft., } 5 \text{ ft.}$$

$$12. \frac{\sqrt[n-1]{a_1 a_2 a_3 \dots a_n}}{a_1}, \frac{\sqrt[n-1]{a_1 a_2 a_3 \dots a_n}}{a_2}, \text{ \&c.}$$

$$13. 2, 3, 6 \text{ days.}$$

14. Solve the equations $P(R^3 - 1) = a$, $PR^3(R - 1) = b$, where P is the principal, and R the amount of £1 for one year.
15. Let x be the equated time, then x is the root of the equation— $240 \left\{ a(m - x) - b(x - n) \right\} = b(x - n)(m - x)$.
16. $x = y = z = \frac{1}{\sqrt{2}}$
17. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

XXIV.—PAGE 129.

1. $\frac{8}{3} \sqrt{165}$. 2. $cx^2 + bx + a = 0$.
3. $(x + \gamma)^2 - p(x + \gamma) + q = 0$. 4. 4.
5. $a(ac' - a'c)^2 + b(a'b - ab')(ac' - a'c) + c(ab - ab')^2 = 0$.
7. $m + n + p = 0$.
8. $x^2 - (p + \sqrt{p^2 - 4q})x + p\sqrt{p^2 - 4q} = 0$.
12. $(3x + 4y + 5)(2x + 3y - 2) = 0$.

XXV.—PAGE 133.

1. a^2 , a^2 , $a^{\frac{2}{3}}$, a . 2. $a + x$, $(a - x)^{\frac{1}{2}}$.
3. $a + a^{\frac{1}{2}}x^{\frac{2}{3}} + x^{\frac{1}{3}}$. 4. $x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
5. $xy^{-1} + x^{-1}y + 1$. 6. $a^2 + ab^{\frac{1}{2}} + b^{\frac{3}{2}}$, $a^{\frac{2}{3}} + b^{\frac{2}{3}}$.
7. $x - x^{\frac{1}{2}}y$. 8. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}}$.
9. $a^{\frac{2}{3}}b^{-\frac{1}{3}} - b^{\frac{2}{3}}$. 13. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}}$.
14. $2x^{\frac{1}{2}}y^{\frac{2}{3}} - 3xy^{\frac{1}{3}} + 2x^{\frac{2}{3}}$. 15. $ab^{-\frac{1}{2}} - 2 + 2a^{-1}b^{\frac{1}{2}}$.
16. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 7$. 17. $x^2y^{-\frac{1}{2}} + 1$. 18. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$.

XXVI.—PAGE 143.

1. $x^{\frac{2}{3}}$, $a^{\frac{2}{3}}b^{\frac{2}{3}}$, $x^{\frac{1}{3}}y^{\frac{2}{3}}$, $a^{\frac{1}{3}}b^{\frac{2}{3}}$.
2. $xya^{-\frac{1}{2}}b^{-\frac{1}{2}}$, $c^{\frac{2}{3}}d^{\frac{1}{3}}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$, $a^{-\frac{n}{m}}x^{\frac{m}{n}}$.

3. $\sqrt{27}, \sqrt{24}, \sqrt[3]{27}, \sqrt[3]{9}$. 4. $\sqrt{32}, \sqrt{27}, \sqrt[3]{32}, \sqrt[3]{27}$.
 5. $\sqrt{9ab}, \sqrt{abc}, \sqrt[3]{a^3 - x^3}$. 6. $\sqrt[3]{4}, \sqrt[3]{27}$.
 7. $\sqrt[12]{8}, \sqrt[12]{81}$. 8. $\sqrt{8}, \sqrt[3]{135}$. 9. $\sqrt[m]{a^n}, \sqrt[m]{b^m}$.
 10. $\sqrt[3]{(a+x)^3}, \sqrt[3]{(a-x)^3}$. 11. $a^{\frac{p+q}{pq}}, b^{\frac{q-p}{pq}}$.
 12. $2\sqrt{3}, 2\sqrt[3]{6}, 6\sqrt{7}, 3\sqrt[3]{3}$.
 13. $2a\sqrt{a+b}, b\sqrt[3]{a^3+b^3}, \frac{1}{4a}\sqrt[3]{a^3+a^2b}$.
 14. $\frac{1}{x} - a, \frac{a^{\frac{3}{2}}}{x^{\frac{3}{2}}} - a^{\frac{3}{2}}x^{\frac{1}{2}}$.
 15. $\frac{x+1}{3x}\sqrt{3x}, \frac{1}{x-a}\sqrt[3]{(x^3-a^3)^3}$.
 16. $4\sqrt{3}, 5\sqrt[3]{7}$. 17. $\frac{ay - ab + c}{b}\sqrt{ab^2x^2}$.
 18. $(3a^{\frac{2}{3}} - 2a^{m+\frac{2}{3}} - 12)\sqrt[3]{a^m}$.
 19. $a^{\frac{3}{2}} - b^{\frac{3}{2}}, a - b$. 20. $(x+y)^{\frac{3}{2}}, a^3 - b^3d^{\frac{3}{2}}$.
 21. $\frac{ac^3}{d} - c\sqrt{\frac{ab^3}{a+b}} + \frac{c^3}{d}\sqrt{\frac{a}{b}(a+b)} - b$.
 22. $a - b^{\frac{3}{2}} + c^{\frac{3}{2}} - d^{\frac{3}{2}} + 2a^{\frac{1}{2}}c^{\frac{1}{2}} - 2b^{\frac{1}{2}}d^{\frac{1}{2}}$.
 23. $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.
 24. $x^{\frac{5}{2}} - x^2y^{\frac{1}{2}} + x^{\frac{3}{2}}y^{\frac{3}{2}} - xy + x^{\frac{1}{2}}y^{\frac{5}{2}} - y^{\frac{5}{2}}$.
 25. $\frac{1}{2}\sqrt{15}, \frac{1}{4}\sqrt{7}, \frac{1}{2}\sqrt[3]{175}$.
 26. $3(2 - \sqrt{3}), \frac{2}{3}(3\sqrt{2} - 2\sqrt{3}), \frac{1}{2}(\sqrt{5} + \sqrt{3})$.
 27. $\frac{2}{3}(2 + \sqrt{2} - \sqrt{6}), \frac{1}{2}(2\sqrt{3} + 3\sqrt{2} - \sqrt{30}), 5 + 2\sqrt{6}$.
 28. $a(x^{\frac{5}{2}} - x^2y^{\frac{1}{2}} + x^{\frac{3}{2}}y^{\frac{3}{2}} - xy + x^{\frac{1}{2}}y^{\frac{5}{2}} - y^{\frac{5}{2}}),$

$$\frac{\frac{1}{2}(3^{\frac{5}{2}} - 3^2 \cdot 2^{\frac{1}{2}} + 3^{\frac{3}{2}} \cdot 2^{\frac{3}{2}} - 3 \cdot 2 + 3^{\frac{1}{2}} \cdot 2^{\frac{5}{2}} - 2^{\frac{5}{2}})}{b}$$

$$\frac{1}{x^2 + xy + y^2}(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y).$$

29. $2 + \sqrt{7}, \sqrt{5} + \sqrt{3}, 5 - \sqrt{5}.$

30. $\sqrt{6} + \sqrt{2}, \sqrt{6} - \sqrt{3}, \sqrt{15} - \sqrt{5}.$

XXVII.—PAGE 148.

1. $a^3 + b^3 : a^3 - b^3$ is the greater, if $a > b$.

2. The former. 3. $b - \frac{ad}{c}.$

4. $(1 - y)(1 + x) : 1 + x^2.$ 5. 30, 35.

6. Less than 5 when x lies between 2 and 3 ; greater than 5 for all values beyond these limits.

8. x satisfies the equation—

$$4x^2 - (a + b - 6c)x + 2(c^2 - ab) = 0.$$

9. $\frac{b}{2}.$ 10. 1. 14. $2b - a, \frac{b^2}{a^2}, \frac{ab}{2a - b}.$

XXVIII.—PAGE 153.

1. 27. 2. 1. 3. 3, 5.

4. $\frac{e, d_1^2}{ed^2}.$ 5. 156 . 8. 6. $s = 16 t^2.$

7. $6\frac{1}{4}$ tons 8. $m^2p : 20n^2q.$

9. $\frac{mc}{m+1} \cdot \frac{n^2}{a^2} \cdot \frac{c}{m+1} \cdot \left(\frac{n}{o}\right)^{\frac{2}{3}}.$ 11. .067.

12. 7.231 ; 23.717.

XXIX.—PAGE 162.

1. 10.

2. $\frac{10}{\frac{2}{2} \frac{2}{2} \frac{3}{3}}.$

3. $\frac{\frac{12}{2} \frac{7}{2}}{\frac{2}{2} \frac{3}{2}}.$

5. $\frac{\frac{9}{2}}{(\frac{2}{2})^3 \frac{3}{3}}.$

6. $\frac{9}{2}$, if no person is to have the same neighbours twice.
 7. 12. 8. 7. 12. m^2 . 16. 10.
 17.
$$\begin{array}{r} 3 \overline{)100} \\ \underline{12} \overline{)88} \end{array}$$

XXX.—PAGE 167.

1. 38, 288. 2. $17\frac{1}{2}$. 3. - 18.
 4. 150. 5. 50^2 . 6. $13a$.
 7. $10x$. 8. $84\frac{1}{2}$. 9. 5, 8, 11, ...
 10. $229\frac{1}{2}$. 11. 2.
 13. $a = \frac{P(1-q) - Q(1-p)}{p-q}$, $d = \frac{P-Q}{p-q}$.
 20. $(3n-7)^2$. 21. 3, 5, 7. 22. 1, 2, 3, 4.
 23. An A. P. where p, q, r are the P^a, Q^a, R^a terms respectively. The common differences are the reciprocals of each other.
 24. 9 or - $10\frac{1}{2}$.

XXXI.—PAGE 172.

1. 96. 2. $\frac{7}{128}$. 3. $60\frac{2}{3}$. 4. $\frac{4}{15} \{ (\frac{5}{2})^{12} - 1 \}$
 5. $\frac{2}{3} \{ 1 - (-\frac{1}{2})^n \}$. 6. $1 - \frac{1}{2^{10}}$ 7. 2.
 8. $\frac{2}{3}$. 9. $5\frac{5}{8}$. 10. $7\frac{1}{2}$. 11. $\frac{6}{11}$.
 12. $\frac{a^2b^2}{ab+1}$. 13. $\frac{(a-b)(a+b)^2}{a+b-1}$. 14. $\frac{1}{2}(11+5\sqrt{5})$.
 15. $2\frac{1}{2}$. 16. $2\frac{2}{3}$. 17. $\frac{1}{2}(\sqrt{3}+1)^2$.

18. $\frac{5}{16}$. 19. $a = -\frac{513}{341}$. 20. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.
 23. $a = \frac{2s_1s_2}{s_1^2 + s_2^2}$, $r = \frac{s_1^2 - s_2^2}{s_1^2 + s_2^2}$. 35. $2(2^n - 1) + \frac{1}{2}n(n+1)$.

XXXII.—PAGE 177.

1. $\frac{1}{8}r$, $\frac{1}{4}r$, $\frac{1}{2}r$, &c. 2. $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, &c.
 3. -6 , ∞ , 6 , 3 and 1 , $\frac{1}{2}$, $\frac{1}{3}$.
 5. The common difference of the A. P. is $\frac{Q-P}{PQ(p-q)}$.
 6. $\frac{2ac - bc - ab}{2b - a - c}$.
 10. $\frac{1}{123}(4 - \sqrt{7})$, $\frac{1}{123}(4 + \sqrt{7})$.

XXXIII.—PAGE 199.

7. $126a^5b^4$, $126a^4b^5$. 8. $252(15x)^5$. 9. $-20a^5b^5$.
 10. $2^n \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n}$. 11. $\frac{135}{8}x^2$, $\frac{45}{32}x^2$.
 12. $-35 \cdot 4^4 \cdot 3^3 a^2b$, $35 \cdot 4^3 \cdot 3^4 \cdot a^{\frac{3}{2}}b^{\frac{1}{2}}$.
 13. $\frac{1}{3^{r+\frac{1}{2}}} \cdot \frac{1 \cdot 6 \cdot 11 \dots (5r-4)}{r} \cdot (5x)^r$.
 14. $\frac{\frac{n}{r}}{\frac{n-r}{r}} a^{n-r} b^r x^{n-2r}$. 15. $a^{-\frac{1}{2}(r+1)} b^{\frac{r}{2}}$.
 21. 1st term. 22. 1st term.

XXXIV.—PAGE 208.

1. $\frac{1}{2}y + \frac{5}{8}y^2 + \frac{15}{8}y^3 + \&c$.
 2. $1 + x - 2x^2 - 5x^4 + \&c$.
 3. $1 + x + \frac{1}{2}x^2 - x^3 + \&c$.
 4. $5x + 5x^2 + 35x^3 + \&c$. 5. $\frac{1}{2}n(n+1)(2n+1)$.

7. $\frac{3}{x+3} - \frac{2}{x+5}$. 8. $\frac{1}{x^2-6} - \frac{1}{x^2+8}$.
9. $\frac{1}{2(x+1)} - \frac{1}{x+2} + \frac{1}{2(x+3)}$.
10. $\frac{1}{x} - \frac{1}{(x+1)^2} - \frac{2}{x+1} + \frac{x}{1+x+x^2}$.
11. $\frac{1}{3x-1} + \frac{2}{x-1} - \frac{1}{x}$.
12. $\frac{1}{1+x} - \frac{x-2}{1-x+x^2}$.
14. $\frac{1}{(a-b)(a-c)(x+a)} + \frac{1}{(b-a)(b-c)(x+b)}$
 $+ \frac{1}{(c-a)(c-b)(x+c)}$.
15. $\frac{a}{(a-b)(c-a)(x+a)} + \frac{b}{(b-c)(a-b)(x+b)}$
 $+ \frac{c}{(c-a)(b-c)(x+c)}$.
16. $\frac{a^2}{(a-b)(a-c)(x+a)} + \frac{b^2}{(b-c)(b-a)(x+b)}$
 $+ \frac{c^2}{(c-a)(c-b)(x+c)}$.
17. $\frac{1}{x+3} - \frac{x}{x^2+3x+7}$.
18. $\frac{a^2 - ma + n}{(a-b)(a-c)(x+a)} + \frac{b^2 - mb + n}{(b-a)(b-c)(x+b)}$
 $+ \frac{c^2 - mc + n}{(c-a)(c-b)(x+c)}$.

XXXV.—PAGE 217.

1. 2, 3, $\frac{1}{2}$, -1, -2, 1.5. 2. $\frac{5}{3}$, 6, 1.5,

3. 1·0791812, 1·5563025, 1·6532125, 1·8750613,
 $\bar{2}$ ·6020600, ·5740313, $\bar{1}$ ·8239087, $\bar{2}$ ·8696761.
 4. 1·3172901, 3·3172901, $\bar{2}$ ·3172901, $\bar{3}$ ·3172901.
 5. 1, $\bar{1}$, $\bar{3}$, $\bar{3}$.
 6. ·020912, 2091200, ·20912, 20912. 7. $\bar{1}$ ·3162760.
 8. 3493·768. 9. ·2427189. 10. 3·1303338.
 11. 6, 5. 12. 1·67. 13. 1·6774495.
 14. 6·7193696. 15. $\bar{1}$ ·8213310. 16. ·0616835.
 17. 2·07892. 18. £177·63 nearly. 19. £663·449.
 20. 2620·248.

XXXVII.—PAGE 236.

1. 16. 2. 111. 3. $\frac{5715}{131072}$.
 4. - 84600. 5. $12 a_0^3 a_1 a_3 + 6 a_0^2 a_2^2 + 12 a_0 a_1^2 a_2 + a_1^4$.
 6. $\frac{1}{17}$. 7. 420. 8. 10.
 9. 840. 10. $6 a_0^2 a_3^3 + 24 a_0 a_1 a_2 a_3 + 4 a_1^3 a_2$.
 11. 0. 12. $\frac{16 \cdot 15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4}$.
 13. $1 + x + \frac{2}{3} x^2 - \frac{2}{3} x^3 - \frac{1}{9} x^4$.
 14. $1 + \frac{7}{3} x^2 + \frac{1}{3} x^3 - \frac{1}{9} x^4$.

XXXVIII.—PAGE 245.

1. £350 (1·035)¹². 2. $n = \frac{\log 2}{\log 1·05}$.
 3. £1250 $\{(1·04)^{10} - 1\}$. 4. $\frac{150·76}{1 - (1·04)^{-10}}$.
 5. 6 $\frac{1}{17}$ months. 6. $n = \frac{\log 5}{\log 1·06}$.

7. $6\frac{1}{2}$. 8. See Ex. 2, page 99.
9. $n = 20 \cdot \frac{\log 6}{\log e}$, reckoning 5 per cent. per annum.
10. $96(1.06)^5$. 11. $\frac{drR^e}{R^e - 1}$.
12. $\frac{\log a - \log(a - br)}{\log(1 + r)}$.

XXXIX.—PAGE 257.

1. $2 + \frac{1}{2+} \frac{1}{3+} \frac{1}{19}$. 5. See Art. 72.
12. $x = 7, 12, 17, \&c.$
 $y = 2, 5, 8, \&c.$

XL.—PAGE 267.

1. $3 + \frac{1}{6+} \frac{1}{6+} \dots$. 2. $4 + \frac{1}{8+} \frac{1}{8+} \dots$.
3. $4 + \frac{1}{1+} \frac{1}{2+} \frac{1}{3+} \dots$. 4. $5 + \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \dots$.
5. $1 + \frac{1}{2+} \frac{1}{1+} \frac{1}{6+} \dots$. 6. $2 + \frac{1}{1+} \frac{1}{8+} \dots$.
7. $3 + \frac{1}{2+} \frac{1}{3+} \frac{1}{10+} \dots$. 8. $3 + \frac{1}{1+} \frac{1}{2+} \dots$.
9. $2 + \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \dots$. 3. $3 + \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \dots$.
10. $1 + \frac{1}{1+} \frac{1}{12+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \dots$.
11. $2 + \frac{1}{3+} \frac{1}{1+} \frac{1}{1+} \frac{1}{3+} \dots$, $\frac{1}{4+} \frac{1}{1+} \frac{1}{1+} \frac{1}{3+} \frac{1}{1+} \dots$.
12. $\frac{1}{1+} \frac{1}{2+} \frac{1}{5+} \frac{1}{2+} \frac{1}{1+} \dots$, $\frac{1}{2+} \frac{1}{5+} \frac{1}{2+} \frac{1}{1+} \frac{1}{2+} \dots$.

$$13. 5 + \frac{1}{1+} \frac{1}{3+} \frac{1}{1+} \frac{1}{3+} \dots,$$

$$1 + \frac{1}{4+} \frac{1}{4+} \frac{1}{1+} \frac{1}{3+} \frac{1}{1+} \dots.$$

$$14. 2a \pm \left(a + \frac{1}{2a + \dots}\right). \quad 15. \sqrt{26}.$$

$$19. \frac{9 + \sqrt{15}}{7}.$$

$$20. 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \&c.; 1, 2, \frac{1}{3}, \frac{1}{4}; 2, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}.$$

$$23. \text{No.}$$

XLII.—PAGE 292.

$$1. \frac{1 + 5x}{1 - 3x + 2x^2}.$$

$$2. \frac{1 + 2x}{1 - x + 2x^2}.$$

$$3. \frac{2 + 5x}{1 - 4x + 7x^2}.$$

$$4. \frac{3 - 7x}{1 - 6x + 3x^2}.$$

$$5. \frac{9 - 2x}{1 + x + 7x^2}.$$

$$6. \frac{6 - 2x + x^2}{1 - 3x + 4x^2 - x^3}.$$

$$7. \Sigma = \frac{3 - 4x}{(1 - 2x)(1 + 5x)}.$$

$$8. \Sigma = \frac{6 - 3x}{(1 + 7x)(1 - 2x)}.$$

$$9. \Sigma = \frac{1 - 2x}{(1 + x)(1 + 2x)(1 - 3x)}.$$

$$10. \Sigma = \frac{1 - 4x}{(1 - 3x)(1 - x)(1 + x)}.$$

$$11. \Sigma = \frac{2 - 3x}{(1 - x)(1 + 5x)}.$$

$$12. \Sigma = \frac{5 + 3x}{(1 - 3x)(1 + 2x)}.$$

$$13. \Sigma = \frac{3 + 5x + 3x^2}{(1 - x)^3}.$$

$$14. \Sigma = \frac{4 + 5x + x^2}{(1 - x)(1 - 2x)(1 - 3x)}.$$

$$15. 1 - 3x + 3x^2 - x^3.$$

$$16. \frac{a - (a - b)x}{(1 - x)^2}.$$

$$19. \frac{1}{4} n(n+1)(n+2)(n+3) + n(n+1)(n+2) + \frac{1}{4} n(n+1).$$

$$20. \frac{1}{4} \{ (x+n)(x+n+1)(x+n+2)(x+n+3) - x(x+1)(x+2)(x+3) \}$$

$$21. \frac{1}{4} \left\{ \frac{23}{120} - \frac{1}{(n+2)(n+4)} - \frac{1}{(n+3)(n+5)} \right\}.$$

$$22. \frac{1}{8} - \frac{3n+5}{(n+2)(n+3)}. \quad 23. \frac{n}{3}(n^2+2).$$

$$24. \frac{1}{2b} \left\{ \frac{1}{(a+b)(a+2b)} - \frac{1}{(a+n+1 \cdot b)(a+n+2 \cdot b)} \right\}.$$

$$25. \frac{n}{3}(4n^2-1). \quad 26. n^2(2n^2-1).$$

$$27. 1 - \frac{1}{n+1}.$$

$$28. 1 + \frac{4}{n+2} - \frac{3}{n+1}. \quad 29. \frac{1}{n}.$$

XLIII.—PAGE 302.

$$1. 3010, 1443.$$

$$2. 744.$$

$$3. 2672.$$

$$4. 9.$$

$$5. 12.$$

$$6. 7.$$

$$7. 117 \cdot 7.$$

$$8. e 41.$$

$$9. e 25 \cdot 33.$$

$$10. 15.$$

$$11. 222.$$

$$12. \frac{1}{2}.$$

$$16. 3^7, 1 \text{ in one scale, and } 3^6, 3^5, 3^4 \text{ in the other.}$$

$$19. p_1^2 = 4 p_0 p_2.$$

XLIV.—PAGE 317.

16. (i.) one; (ii.) equal to G.C.M.; (iii.) may be reduced to (i.) and (ii.).

MISCELLANEOUS EXAMPLES.—PAGE 331.

$$3. \frac{1}{8} (9 \pm \sqrt{833}).$$

$$6. \text{Transpose, cube, \&c.}$$

$$7. 4, 9. \quad 10. \text{According as it is beyond or between 3 and 1;}$$

$$2\sqrt{2} - 3.$$

$$19. (1 + a_0)(1 + a_1) \dots (1 + a_n).$$

$$26. \frac{a^2 - ma + n}{(b - a)(c - a)} \cdot \frac{1}{x - a} + \&c.$$

$$28. \text{The rationalized expression is } a^5 - b^6; 5 - 2\sqrt{3}.$$

$$31. 12789 \cdot 8 \text{ sq. met.} \quad 35. \frac{1}{3+} \frac{1}{3+} \frac{1}{9+} \frac{1}{2+} \dots$$

$$36. 301 \cdot 9. \quad 37. \frac{a}{x}$$

$$43. \frac{B_{2n}}{2^n}, \text{ where } B_1, B_2, \&c., \text{ have the following relation:}$$

$$rB_1 + \frac{r(r-1)}{1 \cdot 2} B_2 + \dots + \frac{r(r-1)}{1 \cdot 2} B_{r-2} + rB_{r-2} = 0.$$

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$$45. x = y + \frac{1}{2}y^3 + \frac{1}{2}y^5 + \frac{1}{15}y^7 + \&c.$$

$$46. x = 8, y = 6. \quad 49. \frac{1}{(a-1)(1+x)}$$

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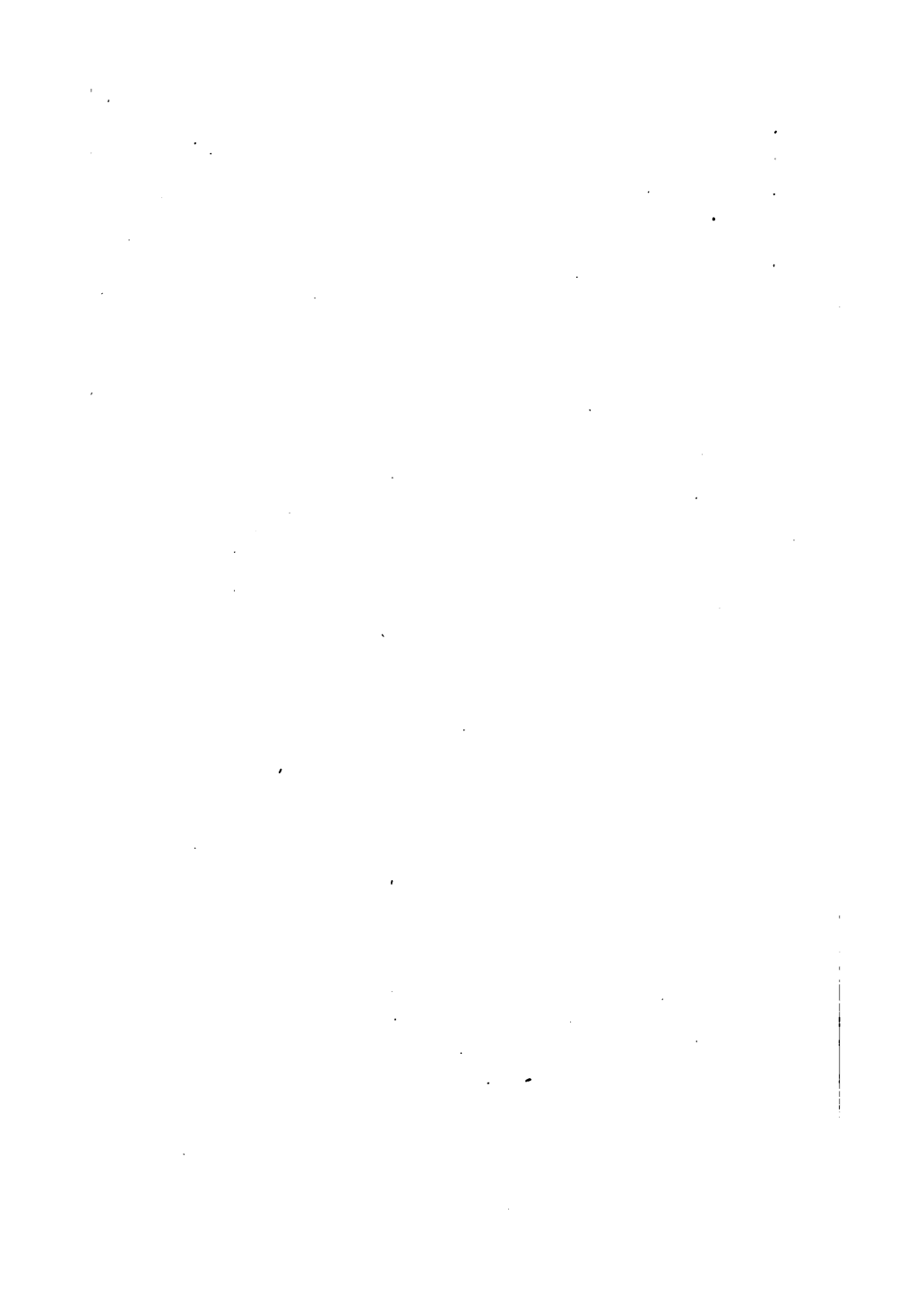
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